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Research Article

# Multivariate Generalized Logistic Neural Networks on Infinite Domain as Positive Linear Operators and a Taste of Measure Theory

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**Abstract:** We study a family of multivariate neural network operators on the infinite domain, constructed via a symmetrized generalized logistic activation function. These operators are treated as positive linear mappings acting on bounded and continuous functions. We establish quantitative approximation results in terms of moduli of continuity and Fréchet derivatives. The operators are also examined within a probabilistic framework, highlighting their connections to measure theory.

**Keywords:** Multivariate neural network operators, quantitative approximation to the unit, infinite domain, probability measure, generalized logistic function.

**AMS Math. Subject Class. (2020):** 41A17, 41A25, 41A36.

## 1 Introduction

The quantitative approximation of positive linear operators to the identity has been a subject of intensive research since the 1980s. The author has made substantial contributions in this area, especially through the introduction of a geometric moment-based framework for analyzing the weak convergence of finite positive measures toward the Dirac measure. This methodology has produced sharp bounds and best-possible estimates, laying the foundation for further developments in the theory of approximation; see [1–4]. These results have been extended to both univariate and multivariate cases, although in the present article we focus solely on the multivariate setting over the infinite domain.

The author has also initiated, since 1997, a systematic investigation into the quantitative convergence of neural network operators to the identity operator. This line of research has resulted in numerous articles and monographs over the years; see [2–6] for representative contributions. The current study draws heavily from the techniques developed in [6] and is particularly guided by the foundational framework outlined in [7].

In this work, we consider for the first time a class of multivariate generalized logistic neural network operators defined on the infinite domain and interpret them within the framework of positive linear operators. This perspective enables the application of classical approximation techniques for positive

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linear operators on unbounded domains, leading to novel and meaningful results concerning both pointwise and uniform convergence.

Convexity cases are also treated here producing sharp results. We also use Fréchet derivatives. Furthermore a strong connection of our operators to measure theory is established with respect to probability measures. The works [8] and [9] have consistently served as key sources of inspiration for the development of the present framework. Further foundational developments in the theory of neural networks can be found in [10–15], which provide a classical background on the subject. For more recent advances and applications in neural network research, the reader is referred to [16–25].

## 2 Fundamental Definitions and Notation

As a starting point, we adopt the approach presented in [6], specifically on pages 395–417. The activation function utilized in our framework is defined via a $q$-deformation and depends on the parameter $\lambda$ as follows:

$$\varphi_{q,\lambda}(x) = \frac{1}{1 + qA^{-\lambda x}}, \quad x \in \mathbb{R}, \ q, \lambda > 0, \ A > 1.$$

The function in question corresponds to the $A$-generalized logistic form. Further discussion can be found in Chapter 16 of [6].

Our present study is inspired by the discussions in Chapters 15 and 16 of [6]. In particular, the “symmetrization technique” we introduce is designed to process only half of the data input in our neural network models. The corresponding density function is given as follows:

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}, \ q, \lambda > 0.$$

We have that

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \tag{2.1}$$

and

$$G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x), \ \forall\, x \in \mathbb{R}. \tag{2.2}$$

By summing equations (2.1) and (2.2), we arrive at the identity

$$G_{q,\lambda}(-x) + G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x), \ \forall\, x \in \mathbb{R},$$

which plays a central role in our analysis. Accordingly, we define

$$W(x) := \frac{G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x)}{2}$$

which is an even function, exhibiting symmetry about the $y$-axis. The global maximum of $G_{q,\lambda}$ occurs at the point given in [6], Eq. (16.18), p. 401:

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^{\lambda} - 1}{2\left(A^{\lambda} + 1\right)}.$$

Likewise, the function $G_{\frac{1}{q},\lambda}$ reaches the same maximal value at the symmetric point

$$G_{\frac{1}{q},\lambda}\left(\frac{\log_A \frac{1}{q}}{\lambda}\right) = G_{\frac{1}{q},\lambda}\left(\frac{-\log_A q}{\lambda}\right) = \frac{A^{\lambda} - 1}{2\left(A^{\lambda} + 1\right)}.$$

Hence, both functions attain identical global maxima at points symmetric with respect to the origin.

From Theorem 16.1, p. 401 of [6], we get

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x-i)=1, \ \ \forall\, x\in\mathbb{R},\ \lambda,q>0,\ \ A>1,$$

and

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x-i)=1, \ \ \forall\, x\in\mathbb{R},\ \lambda,q>0,\ \ A>1.$$

As a direct consequence, we establish the identity

$$\sum_{i=-\infty}^{\infty} W(x-i)=1, \ \ \forall\, x\in\mathbb{R}.$$

Furthermore, based on Theorem 16.2 in [6], p. 402, it follows that

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\,dx=1, \ \ \ \lambda,q>0,\ \ A>1.$$

An analogous result holds for its reciprocal form:

$$\int_{-\infty}^{*} \infty G_{\frac{1}{q},\lambda}(x)\,dx=1.$$

Hence, the average

$$\int_{-\infty}^{\infty} W(x)\,dx=1,$$

confirming that $W(x)$ qualifies as a valid probability density function. In addition, Theorem 16.3 from [6] (see p. 402) asserts that:

Let $0<\alpha<1$ and choose $n\in\mathbb{N}$ such that $n^{1-\alpha}>2$. Under these assumptions, we have the estimate:

$$\sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(nx-k)<2\max\left\{q,\frac{1}{q}\right\}\frac{1}{A^{\lambda\left(n^{1-\alpha}-2\right)}}=\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)},$$

where $\lambda,q>0,\ A>1$, and $\gamma:=2\max\left\{q,\frac{1}{q}\right\}$. A similar inequality holds for the dual function:

$$\sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\alpha}}}^{\infty} G_{\frac{1}{q},\lambda}(nx-k)<\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}.$$

Therefore, for the symmetrized function $W$, we obtain:

$$\sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\alpha}}}^{\infty} W(nx-k)<\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{2.3}$$

confirming the exponential decay of the tail.

We now proceed to the multivariate framework. As a reference model, we follow the exposition in Chapter 15 of [6], pp. 365–394.

**Remark 2.1.** *Let us define the multivariate function*

$$Z(x_1, ..., x_N) := Z(x) := \prod_{i=1}^{N} W(x_i) = \frac{1}{2^N} \prod_{i=1}^{N} \left( G_{q,\lambda} + G_{\frac{1}{q},\lambda} \right)(x_i),$$

$x = (x_1, ..., x_N) \in \mathbb{R}^N$, $\lambda, q > 0$, $N \in \mathbb{N}$. *The function* $Z$ *satisfies the following properties:*

(*i*) *Positivity:*

$$Z(x) > 0, \ \forall x \in \mathbb{R}^N,$$

(*ii*) *Partition of unity:*

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(x_1 - k_1, ..., x_N - k_N) = 1,$$

$k := (k_1, ..., k_n) \in \mathbb{Z}^N$, $\forall \ x \in \mathbb{R}^N$,

*hence*

(*iii*) *Scaled partition of unity:*

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \ \forall x \in \mathbb{R}^N; n \in \mathbb{N}, \tag{2.4}$$

(*iv*) *Normalization:*

$$\int_{\mathbb{R}^N} Z(x)\, dx = 1,$$

*meaning that* $Z$ *is a valid multivariate probability density function.*

(*v*) *Exponential decay estimate: Define the infinity norm by* $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$ *with* $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$. *Then, for any* $0 < \beta < 1$, $n \in \mathbb{N}$ *such that* $n^{1-\beta} > 2$, *we have*

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) < \gamma A^{-\lambda\left(n^{1-\beta}-2\right)}, \tag{2.5}$$

*where* $\gamma := 2\max\left\{q, \frac{1}{q}\right\}$, *and* $x \in \mathbb{R}^N$.

We make

**Remark 2.2.** *Let $x \in [1, \infty)$. Applying the mean value theorem, we obtain*

$$G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(q-1)\right),$$

$$\frac{1}{2}\varphi'_{q,\lambda}(\xi)\, 2 = \varphi'_{q,\lambda}(\xi) = \frac{q\lambda \ln A}{\left(1 + qA^{-\lambda\xi}\right)^2 A^{\lambda\xi}},$$

*where $0 \leq x - 1 < \xi < x + 1$. Then, we drive*

$$G_{q,\lambda}(x) < (q\lambda \ln A)\, A^{-\lambda\xi} < (q\lambda \ln A)\, A^{-\lambda(x-1)}, \quad x \geq 1.$$

*Similarly it holds*

$$G_{\frac{1}{q},\lambda}(x) < \left(\frac{1}{q}\lambda \ln A\right) A^{-\lambda(x-1)}, \quad x \geq 1.$$

*And finally, it holds*

$$W(x) < \frac{1}{2}\left(q + \frac{1}{q}\right)(\lambda \ln A)\, A^{-\lambda(x-1)}, \quad x \geq 1, \tag{2.6}$$

*where $q, \lambda > 0,\ A > 1$.*

We also make

**Remark 2.3.** *We would like to estimate $(n, m \in \mathbb{N})$ :*

$$\begin{aligned}
&\sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} W(|nx-k|)\, |nx-k|^m \\
\overset{(2.6)}{\leq}\ & \frac{1}{2}\left(q + \frac{1}{q}\right)(\lambda \ln A) \sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} |nx-k|^m A^{-\lambda(|nx-k|-1)} \\
=\ & \frac{1}{2}\left(q + \frac{1}{q}\right)(\lambda \ln A) \sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} |nx-k|^m e^{-\lambda(\ln A)(|nx-k|-1)} \\
=\ & \frac{1}{2}\left(q + \frac{1}{q}\right)\lambda(\ln A)\, A^{\lambda} \sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} |nx-k|^m e^{-\lambda(\ln A)|nx-k|} \\
=:\ & (*)\,.
\end{aligned}$$

*For convenience set $\mu := \lambda \ln A > 0$. Notice that*

$$e^{\frac{\mu|nx-k|}{2}} = \sum_{\theta=0}^{\infty} \frac{\left(\frac{\mu|nx-k|}{2}\right)^{\theta}}{\theta!} \geq \left(\frac{\mu\,|nx-k|}{2}\right)^m \frac{1}{m!}.$$

*Then, we get*

$$\left(\frac{\mu\,|nx-k|}{2}\right)^m \leq m! e^{\frac{\mu|nx-k|}{2}}, \quad \text{or}$$

$$(\mu |nx-k|)^m \leq 2^m m! e^{\frac{\mu|nx-k|}{2}}, \text{ or}$$
$$|nx-k|^m \leq \frac{2^m}{\mu^m} m! e^{\frac{\mu|nx-k|}{2}}.$$

*Therefore, we see that*

$$(*) \leq \frac{1}{2}\left(q+\frac{1}{q}\right)\lambda(\ln A) A^{\lambda}$$

$$\begin{aligned}
& \sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\beta}}}^{\infty} \frac{2^m}{\lambda^m (\ln A)^m} m! e^{\frac{\lambda \ln A |nx-k|}{2}} e^{-\lambda \ln A |nx-k|} \\
= & \frac{1}{2}\left(q+\frac{1}{q}\right)\lambda(\ln A) A^{\lambda} \frac{2^m}{\lambda^m (\ln A)^m} m! \sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\beta}}}^{\infty} e^{-\frac{\lambda \ln A|nx-k|}{2}} \\
= & \frac{1}{2}\left(q+\frac{1}{q}\right)\frac{A^{\lambda}2^m m!}{\lambda^{m-1}(\ln A)^{m-1}}\left(\sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\beta}}}^{\infty} e^{-\frac{\lambda \ln A}{2}|nx-k|}\right) \\
\leq & \left(q+\frac{1}{q}\right)\frac{A^{\lambda}2^m m!}{\lambda^{m-1}(\ln A)^{m-1}}\left(\int_{n^{1-\beta}-1}^{\infty} e^{-\frac{\lambda \ln A}{2}x} dx\right) \\
= & \frac{2}{\lambda \ln A}\left(q+\frac{1}{q}\right)\frac{A^{\lambda}2^m m!}{\lambda^{m-1}(\ln A)^{m-1}}\left(\int_{n^{1-\beta}-1}^{\infty} e^{-\frac{\lambda \ln A}{2}x} d\left(\frac{\lambda \ln A}{2}\right)x\right) \\
= & \frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^m(\ln A)^m}\left(\int_{n^{1-\beta}-1}^{\infty} e^{-y}dy\right) \\
= & \frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^m(\ln A)^m}\left(-e^{-y}|_{n^{1-\beta}-1}^{\infty}\right) \\
= & \frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^m(\ln A)^m}\left(e^{-y}|_{\infty}^{n^{1-\beta}-1}\right) \\
= & \frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^m(\ln A)^m}\left(e^{-\frac{\lambda \ln A}{2}x}|_{\infty}^{n^{1-\beta}-1}\right) \\
= & \frac{\left(q+\frac{1}{q}\right)A^{\lambda}2^{m+1}m!}{\lambda^m(\ln A)^m}\left(e^{-\frac{\lambda \ln A}{2}\left(n^{1-\beta}-1\right)}\right).
\end{aligned}$$

*Thus, we have proved:*

$$\sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\beta}}}^{\infty} W(|nx-k|)|nx-k|^m \leq \frac{\left(q+\frac{1}{q}\right)2^{m+1}m!}{\lambda^m(\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}, \tag{2.7}$$

*where* $m, n \in \mathbb{N}$; $q, \lambda > 0$, $A > 1$, $0 < \beta < 1$; *see also* [7].

We also make

**Remark 2.4.** *We also estimate* ($\gamma := 2\max\left\{q, \frac{1}{q}\right\}$)

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} W(|nx-k|)(1+|nx-k|)^m$$

$$\leq 2^{m-1} \sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} W(|nx-k|)(1+|nx-k|^m)$$

$$= 2^{m-1} \left\{ \sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} W(|nx-k|) + \sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} W(|nx-k|)\,|nx-k|^m \right\}$$

$$\overset{(\text{by } (2.3),(2.7))}{\leq} 2^{m-1} \left\{ \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \left( \frac{\left(q+\frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} \right) A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right\}.$$

*We have proved that*

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} W(|nx-k|)(1+|nx-k|)^m$$

$$\leq 2^{m-1} \left\{ \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \left( \frac{\left(q+\frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} \right) A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right\},$$

*where* $m, n \in \mathbb{N}$; $q, \lambda > 0$, $A > 1$, $0 < \beta < 1$, $\gamma = 2\max\left\{q, \frac{1}{q}\right\}$.

We give

**Theorem 2.1.** *Let* $0 < \beta < 1$ *and choose* $n \in \mathbb{N}$ *such that* $n^{1-\beta} > 2$, $m \in \mathbb{N}$; $A > 1$, $q, \lambda > 0$, $\gamma = 2\max\left\{q, \frac{1}{q}\right\}$. *Then*

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ \left\{k \in \mathbb{Z}^N : \|nx-k\|_\infty \geq n^{1-\beta}\right\} \end{array}\right.}^{\infty} \|nx-k\|_\infty^m Z(nx-k) < \frac{\left(q+\frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}. \quad (2.8)$$

*Proof.* Assume that $k \in \mathbb{Z}^N$ satisfies the condition $\|nx-k\|_\infty \geq n^{1-\beta}$. By the definition of the infinity norm, this implies that there exists at least one index $r \in \{1,...,N\}$ such that the inequality $|nx_r - k_r| \geq n^{1-\beta}$ holds for some $k_r \in \mathbb{Z}$. In other words, the set of all such multi-indices $k$ is contained in a union of sets, each characterized by a deviation in a single coordinate direction. This allows us to reduce the multidimensional analysis to a one-dimensional estimate along the direction in which the deviation occurs. Then, we get

$$\begin{aligned} & \left\{k \in \mathbb{Z}^N : \|nx-k\|_\infty \geq n^{1-\beta}\right\} \\ \subset\ & \cup_{r=1}^N \left\{\mathbb{Z}^{N-1} \cup \left\{k_r \in \mathbb{Z} : |nx_r - k_r| \geq n^{1-\beta}\right\}\right\} \\ \subset\ & \mathbb{Z}^{N-1} \cup \left\{k_{r^*} \in \mathbb{Z} : |nx_{r^*} - k_{r^*}| \geq n^{1-\beta}\right\}, \end{aligned}$$

for some $k^* \in \{1,...,N\}$. Next, we observe that

$$\begin{aligned} & \sum_{\substack{k=-\infty \\ \left\{k \in \mathbb{Z}^N : \|nx-k\|_\infty \geq n^{1-\beta}\right\}}}^{\infty} \|nx-k\|_\infty^m Z(nx-k) \\ \leq\ & \sum_{\substack{k_{r^*}=-\infty \\ \left\{k_{r^*} \in \mathbb{Z}^N : |nx_{r^*} - k_{r^*}| \geq n^{1-\beta}\right\}}}^{\infty} |nx_{r^*} - k_{r^*}|^m W(|nx_{r^*} - k_{r^*}|) \\ <\ & \frac{\left(q+\frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}, \end{aligned}$$

where we used (2.7). □

We also give

**Theorem 2.2.** *All as in Theorem 2.1. Then*

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} Z(nx-k)\left(1+\|nx-k\|_\infty\right)^m \\ \leq\ & 2^{m-1}\left(\gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \frac{\left(q+\frac{1}{q}\right)2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right). \end{aligned} \tag{2.9}$$

*Proof.* We have

$$
\begin{aligned}
&\sum_{\begin{cases} k=-\infty \\ \|nx-k\|_{\infty}\geq n^{1-\beta} \end{cases}}^{\infty} Z(nx-k)\left(1+\|nx-k\|_{\infty}\right)^{m} \\
\leq\ & 2^{m-1}\left(\sum_{\begin{cases} k=-\infty \\ \|nx-k\|_{\infty}\geq n^{1-\beta} \end{cases}}^{\infty} Z(nx-k)\left(1+\|nx-k\|_{\infty}^{m}\right)\right) \\
=\ & 2^{m-1}\left(\sum_{\begin{cases} k=-\infty \\ \|nx-k\|_{\infty}\geq n^{1-\beta} \end{cases}}^{\infty} Z(nx-k)\right. \\
+\ & \left.\sum_{\begin{cases} k=-\infty \\ \|nx-k\|_{\infty}\geq n^{1-\beta} \end{cases}}^{\infty} Z(nx-k)\|nx-k\|_{\infty}^{m}\right),
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\sum_{\begin{cases} k=-\infty \\ \|nx-k\|_{\infty}\geq n^{1-\beta} \end{cases}}^{\infty} Z(nx-k)\left(1+\|nx-k\|_{\infty}\right)^{m} \\
\leq\ & 2^{m-1}\left(\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}+\frac{\left(q+\frac{1}{q}\right)2^{m+1}m!}{\lambda^{m}(\ln A)^{m}}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right).
\end{aligned}
$$

due to (2.5) and (2.8).

□

**Definition 2.1.** *Let $f:\mathbb{R}^N\to\mathbb{R}$ be a bounded and continuous function, that is, $f\in C_B\left(\mathbb{R}^N\right)$, where $N\in\mathbb{N}$. The modulus of continuity, denoted by $\omega_1(f,\delta)$, is defined for any $\delta>0$ as*

$$
\omega_1(f,\delta):=\sup_{\substack{x,y\in\mathbb{R}^N:\\ \|x-y\|_{\infty}<\delta}}|f(x)-f(y)|,\quad \delta>0,
$$

*In a similar fashion, the modulus $\omega_1$ can also be defined for functions in the space $C_U\left(\mathbb{R}^N\right)$, that is, for all bounded uniformly continuous functions. A function $f$ belongs to $C_U\left(\mathbb{R}^N\right)$ if and only if $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$. For further details, see [5].*

Denote by $B\left(\mathbb{R}^N\right)$ the bounded functions on $\mathbb{R}^N$. Our attention now turns to the definition of the neural network operators.

**Definition 2.2.** *Let $f \in C_B\left(\mathbb{R}^N\right)$. We define the multivariate quasi-interpolation neural network operator $\widetilde{A}_n(f,x)$ by*

$$\widetilde{A}_n(f,x) := \widetilde{A}_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k),$$

*where $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Equivalently, this can be written as*

$$\widetilde{A}_n(f,x) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} W(nx_i-k_i)\right)$$

*In addition, for any $f \in C_B\left(\mathbb{R}^N\right)$, we also define the multivariate Kantorovich-type neural network operator as follows:*

$$\begin{aligned} & \widetilde{K}_n(f,x) := \widetilde{K}_n(f,x_1,...,x_N) \\ := & \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z(nx-k) \\ := & \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,...,t_N)\,dt_1...dt_N\right) \\ \times & \left(\prod_{i=1}^{N} W(nx_i-k_i)\right) \end{aligned}$$

*for $n \in \mathbb{N}$ and $\forall\ x \in \mathbb{R}^N$. Again for $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $\widetilde{Q}_n(f,x)$, $n \in \mathbb{N}$, as follows. Let $\theta = (\theta_1,...,\theta_N) \in \mathbb{N}^N$, $r = (r_1,...,r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1 r_2 ... r_N} \geq 0$, such that*

$$\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1 r_2 ... r_N} = 1; \quad k \in Z^N,$$

*and*

$$\begin{aligned} & \delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) \\ := & \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1 r_2 ... r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \end{aligned}$$

*where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$. We put*

$$\begin{aligned} & \widetilde{Q}_n(f,x) := \widetilde{Q}_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx-k) \\ := & \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,k_2,...,k_N}(f) \left(\prod_{i=1}^{N} W(nx_i-k_i)\right), \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

**Remark 2.5.** *We notice that*

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, \dots, t_N)\, dt_1 dt_2 \dots dt_N$$

$$= \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \dots, t_N + \frac{k_N}{n}\right) dt_1 \dots dt_N$$

$$= \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt.$$

*Thus, it holds*

$$\widetilde{K}_n(f, x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) Z(nx - k).$$

*We do have that* $\widetilde{A}_n(1) = \widetilde{K}_n(1) = \widetilde{Q}_n(1) = 1$, $\forall\, n \in \mathbb{N}$. *Denote by* $\widetilde{\Theta}_n$ *any of* $\widetilde{A}_n$, $\widetilde{K}_n$, $\widetilde{Q}_n$ *and they are positive linear operators mapping* $C_B(\mathbb{R}^N)$ *into* $B(\mathbb{R}^N)$.

## 3 Auxilliary Results

Here let $x_0 \in \mathbb{R}^N$, $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$ and $m \in \mathbb{N}$. It follows:

**Proposition 3.1.** *It holds that*

$$\widetilde{A}_n\left(\|\cdot - x_0\|_\infty^m\right)(x_0) > 0,$$

*and*

$$0 < \left\|\widetilde{A}_n\left(\|\cdot - x_0\|_\infty^m\right)(x_0)\right\|_\infty \leq \left(\frac{1}{n^\beta} + \frac{1}{n^m} \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta} - 3\right)}\right) < \infty \tag{3.1}$$

($\to 0$, *as* $n \to +\infty$).

*Proof.* We have that

$$0 < \widetilde{A}_n\left(\|\cdot - x_0\|_\infty^m\right)(x_0) = \sum_{k=-\infty}^{\infty} \left\|\frac{k}{n} - x_0\right\|_\infty^m Z(nx_0 - k)$$

$$= \sum_{\substack{k = -\infty \\ \left\|\frac{k}{n} - x_0\right\|_\infty < \frac{1}{n^\beta}}}^{\infty} \left\|\frac{k}{n} - x_0\right\|_\infty^m Z(nx_0 - k)$$

$$+ \sum_{\substack{k = -\infty \\ \left\|\frac{k}{n} - x_0\right\|_\infty \geq \frac{1}{n^\beta}}}^{\infty} \left\|\frac{k}{n} - x_0\right\|_\infty^m Z(nx_0 - k)$$

$$\leq \frac{1}{n^\beta} + \frac{1}{n^m} \sum_{\substack{k = -\infty \\ \|nx_0 - k\|_\infty \geq n^{1-\beta}}}^{\infty} \|nx_0 - k\|_\infty^m Z(nx_0 - k)$$

$$\overset{(2.8)}{\leq} \frac{1}{n^\beta} + \frac{1}{n^m} \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta} - 3\right)}.$$

Therefore it holds

$$0 < \left\| \widetilde{A}_n \left( \|\cdot - x_0\|_\infty^m \right)(x_0) \right\|_\infty \leq \frac{1}{n^\beta} + \frac{1}{n^m} \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \to 0, \quad \text{as } n \to +\infty.$$

□

**Proposition 3.2.** *It holds*

$$\begin{aligned} &0 < \widetilde{K}_n \left( \|\cdot - x_0\|_\infty^m \right)(x_0) \\ \leq\ & \left[ \left( \frac{1}{n} + \frac{1}{n^\beta} \right)^m + \frac{2^{m-1}}{n^m} \left[ \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right] \right] < \infty, \end{aligned}$$

$\forall\ x_0 \in \mathbb{R}^N$. *($\to 0$, as $n \to +\infty$). That is*

$$\begin{aligned} &0 < \left\| \widetilde{K}_n \left( \|\cdot - x_0\|_\infty^m \right)(x_0) \right\|_\infty \\ \leq\ & \left[ \left( \frac{1}{n} + \frac{1}{n^\beta} \right)^m + \frac{2^{m-1}}{n^m} \left[ \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right] \right] < \infty, \end{aligned}$$

*($\to 0$, as $n \to +\infty$).*

*Proof.* We have that

$$\begin{aligned} &0 < \widetilde{K}_n \left( \|\cdot - x_0\|_\infty^m \right)(x_0) = \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} \left\| t + \frac{k}{n} - x_0 \right\|_\infty^m dt \right) Z(nx_0 - k) \\ \leq\ & \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} \left( \left\| \frac{k}{n} - x_0 \right\|_\infty + \|t\|_\infty \right)^m dt \right) Z(nx_0 - k) \\ \leq\ & \sum_{k=-\infty}^{\infty} Z(nx_0 - k) \left( \left\| \frac{k}{n} - x_0 \right\|_\infty + \frac{1}{n} \right)^m \\ =\ & \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x_0 \right\|_\infty < \frac{1}{n^\beta}}}^{\infty} Z(nx_0 - k) \left( \left\| \frac{k}{n} - x_0 \right\|_\infty + \frac{1}{n} \right)^m \\ +\ & \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x_0 \right\|_\infty \geq \frac{1}{n^\beta}}}^{\infty} Z(nx_0 - k) \left( \left\| \frac{k}{n} - x_0 \right\|_\infty + \frac{1}{n} \right)^m \\ \leq\ & \left( \frac{1}{n^\beta} + \frac{1}{n} \right)^m + \frac{1}{n^m} \left( \sum_{\substack{k=-\infty \\ \|nx_0 - k\|_\infty \geq n^{1-\beta}}}^{\infty} Z(nx_0 - k) \left( \|nx_0 - k\|_\infty + 1 \right)^m \right) \\ \overset{(2.9)}{\leq}\ & \left( \frac{1}{n^\beta} + \frac{1}{n} \right)^m + \frac{2^{m-1}}{n^m} \left( \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right). \end{aligned}$$

□

**Proposition 3.3.** *It holds that*

$$0 < \widetilde{Q}_n \left( \|\cdot - x_0\|_\infty^m \right) (x_0) \leq \left( \frac{1}{n} + \frac{1}{n^\beta} \right)^m + \frac{2^{m-1}}{n^m} \left[ \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right] < \infty,$$

$\forall\ x_0 \in \mathbb{R}^N$, *($\to 0$, as $n \to +\infty$). Hence we get*

$$0 < \left\| \widetilde{Q}_n \left( \|\cdot - x_0\|_\infty^m \right) (x_0) \right\|_\infty \leq \left( \frac{1}{n} + \frac{1}{n^\beta} \right)^m + \frac{2^{m-1}}{n^m} \left[ \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right] < \infty,$$

*($\to 0$, as $n \to +\infty$).*

*Proof.* We notice that

$$\begin{aligned}
\widetilde{Q}_n \left( \|\cdot - x_0\|_\infty^m \right) (x_0) &= \sum_{k=-\infty}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left\| \frac{k}{n} + \frac{r}{n\theta} - x_0 \right\|_\infty^m \right) Z(nx_0 - k) \\
&\leq \sum_{k=-\infty}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left( \left\| \frac{k}{n} - x_0 \right\|_\infty + \left\| \frac{r}{n\theta} \right\|_\infty \right)^m \right) Z(nx_0 - k) \\
&\leq \sum_{k=-\infty}^{\infty} \left( \left\| \frac{k}{n} - x_0 \right\|_\infty + \frac{1}{n} \right)^m Z(nx_0 - k) \\
&\leq \left( \frac{1}{n} + \frac{1}{n^\beta} \right)^m + \frac{2^{m-1}}{n^m} \left[ \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \frac{\left(q + \frac{1}{q}\right) 2^{m+1} m!}{\lambda^m (\ln A)^m} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right] < \infty
\end{aligned}$$

as earlier in Proposition 3.2. □

## 4 Main Results

Next we present approximation results.

**Theorem 4.1.** *Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$. It holds*

$$\left\| \widetilde{A}_n (f) - f \right\|_\infty \leq 2\omega_1 \left( f, \left( \frac{1}{n^\beta} + \frac{1}{n} \frac{\left(q + \frac{1}{q}\right) 4}{\lambda \ln A} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right) \right) < \infty.$$

*Futrhermore, if $f$ is uniformly continuous, then we observe that $\widetilde{A}_n (f) \to f$, pointwise and uniformly, as $n \to +\infty$.*

*Proof.* We can write

$$\widetilde{A}_n (f, x) = \int_{\mathbb{R}^N} f(t)\, d\mu_x (t),$$

where $\mu_x\left(\mathbb{R}^N\right)=1$, $\mu_x$ is a probability measure on $\mathbb{R}^N$. The support of $\mu_x$ is discrete and countable, at the points $\frac{k}{n}=\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\in\mathbb{R}^N$, $k_i\in\mathbb{Z}$, $i=1,...,N$. Each $\frac{k}{n}$ carries the weight $Z(nx-k)=\prod_{i=1}^{N}W(nx_i-k_i)$. Indeed here we have that

$$\sum_{k=-\infty}^{\infty}Z(nx-k)\overset{(2.4)}{=}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\left(\prod_{i=1}^{N}W(nx_i-k_i)\right)=1.$$

Here it is $\omega_1(f,h)<\infty$, $h>0$. By Lemma 7.1.1, p. 208, [4], we get that

$$|f(t)-f(x)|\leq\omega_1(f,h)\left\lceil\frac{\|t-x\|_\infty}{h}\right\rceil,$$

where $\lceil\cdot\rceil$ is the ceiling function. But it holds

$$\left\lceil\frac{\|t-x\|_\infty}{h}\right\rceil\leq 1+\frac{\|t-x\|_\infty}{h}.$$

Thus we get

$$|f(t)-f(x)|\leq\omega_1(f,h)\left[1+\frac{\|t-x\|_\infty}{h}\right],\quad\forall\, t\in\mathbb{R}^N.$$

The last can be written also as

$$|f(\cdot)-f(x)|\leq\omega_1(f,h)\left[1+\frac{\|\cdot-x\|_\infty}{h}\right],\quad\text{over }\mathbb{R}^N.$$

We have that

$$\widetilde{A}_n(f,x)-f(x)=\int_{\mathbb{R}^N}f(t)\,d\mu_x(t)-f(x)=\int_{\mathbb{R}^N}(f(t)-f(x))\,d\mu_x(t).$$

Therefore

$$\left|\widetilde{A}_n(f,x)-f(x)\right|\leq\left|\int_{\mathbb{R}^N}(f(t)-f(x))\,d\mu_x(t)\right|\leq$$

$$\int_{\mathbb{R}^N}|f(t)-f(x)|\,d\mu_x(t)\leq\omega_1(f,h)\left[1+\frac{\int_{\mathbb{R}^N}\|t-x\|_\infty\,d\mu_x(t)}{h}\right].$$

Choosing $h:=\int_{\mathbb{R}^N}\|t-x\|_\infty\,d\mu_x(t)$, we obtain

$$\left|\widetilde{A}_n(f,x)-f(x)\right|\leq 2\omega_1\left(f,\int_{\mathbb{R}^N}\|t-x\|_\infty\,d\mu_x(t)\right)=2\omega_1\left(f,\widetilde{A}_n\left(\|\cdot-x\|_\infty\right)(x)\right).$$

But we have $(0 < \beta < 1)$

$$0 < \widetilde{A}_n \left( \|\cdot - x\|_\infty \right)(x) = \sum_{k=-\infty}^{\infty} \left\| \frac{k}{n} - x \right\|_\infty Z(nx - k)$$

$$= \sum_{\substack{k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty < \frac{1}{n^\beta}}}^{\infty} \left\| \frac{k}{n} - x \right\|_\infty Z(nx - k)$$

$$+ \sum_{\substack{k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \geq \frac{1}{n^\beta}}}^{\infty} \left\| \frac{k}{n} - x \right\|_\infty Z(nx - k)$$

$$\leq \frac{1}{n^\beta} + \frac{1}{n} \sum_{\substack{k = -\infty \\ \|nx - k\|_\infty \geq n^{1-\beta}}}^{\infty} \|nx - k\|_\infty Z(nx - k)$$

$$\overset{(2.8)}{\leq} \frac{1}{n^\beta} + \frac{1}{n} \frac{\left(q + \frac{1}{q}\right) 4}{\lambda \ln A} A^{-\frac{\lambda}{2}\left(n^{1-\beta} - 3\right)}.$$

So that it holds

$$\widetilde{A}_n \left( \|\cdot - x\|_\infty \right)(x) \leq \frac{1}{n^\beta} + \frac{1}{n} \frac{\left(q + \frac{1}{q}\right) 4}{\lambda \ln A} A^{-\frac{\lambda}{2}\left(n^{1-\beta} - 3\right)} (< +\infty) \to 0,$$

as $n \to +\infty$. Tehn, we get

$$\left| \widetilde{A}_n (f, x) - f(x) \right| \leq 2\omega_1 \left( f, \left( \frac{1}{n^\beta} + \frac{1}{n} \frac{\left(q + \frac{1}{q}\right) 4}{\lambda \ln A} A^{-\frac{\lambda}{2}\left(n^{1-\beta} - 3\right)} \right) \right) < \infty,$$

which completes the proof. ☐

Next comes a convexity approximation result.

**Theorem 4.2.** *Let $f \in C_B\left(\mathbb{R}^N\right) : |f(\cdot) - f(x_0)|$ is convex, $x_0 \in \mathbb{R}^N$. Then*

$$\left| \widetilde{A}_n (f, x_0) - f(x_0) \right| \leq \omega_1 \left( f, \left( \frac{1}{n^\beta} + \frac{1}{n} \frac{\left(q + \frac{1}{q}\right) 4}{\lambda \ln A} A^{-\frac{\lambda}{2}\left(n^{1-\beta} - 3\right)} \right) \right) < +\infty,$$

*($\to 0$, as $n \to \infty$, given that $f$ is uniformly continuous, i.e. $\lim_{n \to +\infty} \widetilde{A}_n (f, x_0) = f(x_0)$).*

*Proof.* Based on Lemma 8.1.1, p. 243, [4], we obtain the following. Let $(V, \|\cdot\|)$ be a real normed vector space and $f : V \to \mathbb{R}$ such that $|f(t) - f(x_0)|$ is convex in $t \in V$; $x_0 \in V$. Then

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f, h)}{h} \|t - x_0\|, \quad \forall\, t \in V,\ h > 0.$$

That is, it holds here

$$|f(\cdot) - f(x_0)| \leq \frac{\omega_1(f, h)}{h} \|\cdot - x_0\|_\infty, \text{ over } \mathbb{R}^N. \tag{4.1}$$

We have that

$$\widetilde{A}_n(f,x_0)-f(x_0)=\int_{\mathbb{R}^N}(f(t)-f(x_0))\,d\mu_{x_0}(t),$$

and

$$\left|\widetilde{A}_n(f,x_0)-f(x_0)\right|\leq\int_{\mathbb{R}^N}|f(t)-f(x_0)|\,d\mu_{x_0}(t)\leq\frac{\omega_1(f,h)}{h}\int_{\mathbb{R}^N}\|t-x_0\|_\infty\,d\mu_{x_0}(t)$$

(we take $h:=\int_{\mathbb{R}^N}\|t-x_0\|_\infty\,d\mu_{x_0}(t)=\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)>0$)

$$=\omega_1\left(f,\int_{\mathbb{R}^N}\|t-x_0\|_\infty\,d\mu_{x_0}(t)\right)=\omega_1\left(f,\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)\right).$$

Thus, it holds

$$\left|\widetilde{A}_n(f,x_0)-f(x_0)\right|\leq\omega_1\left(f,\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)\right). \tag{4.2}$$

Inequality (4.2) is sharp, namely it is attained by $f(t)=\|t-x_0\|_\infty$, $\forall\, t\in\mathbb{R}^N$, $x_0\in\mathbb{R}^N$. We have

$$\left|\widetilde{A}_n(f,x_0)-f(x_0)\right|=\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0).$$

Next we observe

$$\begin{aligned}
&\omega_1\left(f,\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)\right)=\omega_1\left(\|\cdot-x_0\|_\infty,\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)\right)\\
=&\sup_{\left\{\begin{array}{c}t_1,t_2\in\mathbb{R}^N:\\ \|t_1-t_2\|_\infty\leq\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)\end{array}\right.}\left|\|t_1-x_0\|_\infty-\|t_2-x_0\|_\infty\right|\\
\leq&\sup_{\left\{\begin{array}{c}t_1,t_2\in\mathbb{R}^N:\\ \|t_1-t_2\|_\infty\leq\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)\end{array}\right.}\|t_1-t_2\|_\infty=\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0).
\end{aligned}$$

So that

$$\omega_1\left(f,\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)\right)=\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0).$$

Thus (4.2) it is attained. Continuing on (4.2) we have that

$$\omega_1\left(f,\widetilde{A}_n(\|\cdot-x_0\|_\infty)(x_0)\right)\leq\omega_1\left(f,\left(\frac{1}{n^\beta}+\frac{1}{n}\frac{\left(q+\frac{1}{q}\right)4}{\lambda\ln A}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right)\right).$$

The theorem is proved. ∎

From [4], pp. 232-233 we mention

**Theorem 4.3.** *Let $\mu$ be a finite measure of mass $m>0$ on $M\neq\varnothing$, which is a convex subset of the Banach space $(V,\|\cdot\|)$ and $x_0\in M$ be given. Assume that*

$$\left(\frac{1}{m}\int_M\|x-x_0\|^{n+1}\mu(dx)\right)^{\frac{1}{(n+1)}}:=D_{n+1}(x_0)>0,\quad n\in\mathbb{N}.$$

*Let $f : M \to \mathbb{R}$ be a bounded continuous function, whose Fréchet derivatives $f^{(j)}$, $1 \leq j \leq n$ exist, are bounded and continuous functions. Assume that $\omega_1\left(f^{(n)}, rD_{n+1}(x_0)\right) \leq w$, where $r, w > 0$ are given. Then*

$$\begin{aligned}\left|\int_M f d\mu - f(x_0)\right| &\leq |f(x_0)|\,|m-1| + \left|\sum_{j=1}^{n} \frac{1}{j!}\int_M f^{(j)}(x_0)(x-x_0)^j \mu(dx)\right| \\ &+ \frac{mw}{rn!}\left[\frac{nr^2}{8} + \frac{r}{2} + \frac{1}{(n+1)}\right] D_{n+1}^n(x_0).\end{aligned}$$

**Corollary 4.1.** *When $\mu(M) = 1$ and $\left(\int_M \|x-x_0\|^2 \mu(dx)\right)^{\frac{1}{2}} := D_2(x_0) > 0$, it holds:*

$$\left|\int_M f d\mu - f(x_0)\right| - \left|\int_M f^{(1)}(x_0)(x-x_0)\mu(dx)\right| \leq (1.5625)\,\omega_1\left(f^{(1)}, \frac{1}{2}D_2(x_0)\right) D_2(x_0).$$

**Proposition 4.1.** *Let $\mu$ be a finite measure of mass $m > 0$ on $M \neq \varnothing$, which is a convex subset of the Banach space $(V, \|\cdot\|)$ and $x_0 \in M$ be given. Assume that*

$$\left(\int_M \|x-x_0\|^2 \mu(dx)\right)^{\frac{1}{2}} := D_2(x_0) > 0$$

*and consider $r > 0$. Let $f : M \to \mathbb{R}$ be a bounded continuous function, whose Fréchet derivative $f^{(1)}$ exists, is bounded and continuous. Then*

$$\begin{aligned}&\left|\int_M f d\mu - f(x_0)\right| - |f(x_0)|\,|m-1| - \left|\int_M f^{(1)}(x_0)(x-x_0)\mu(dx)\right| \\ \leq\ &\frac{1}{8r}\left(2+\sqrt{m}r\right)^2 \omega_1\left(f', rD_2(x_0)\right) D_2(x_0).\end{aligned}$$

**Corollary 4.2.** *Let $\mu$ be a finite measure of mass $m > 0$ on $M$, which is a convex subset of the Banach space $(V, \|\cdot\|)$ and $x_0 \in M$ be given. Assume that $\left(\int_M \|x-x_0\|^2 \mu(dx)\right)^{\frac{1}{2}} := D_2(x_0) > 0$. Consider $r > 0$, let $f : M \to \mathbb{R}$ be a bounded and continuous function, whose Fréchet derivative $f^{(1)}$ exists, is bounded and continuous. Then*

$$\begin{aligned}&\left|\int_M f d\mu - f(x_0)\right| - |f(x_0)|\,|m-1| - \left|\int_M f^{(1)}(x_0)(x-x_0)\mu(dx)\right| \\ \leq\ &\begin{cases} \frac{1}{8r}\left(2+\sqrt{m}r\right)^2 \omega_1\left(f^{(1)}, rD_2(x_0)\right) D_2(x_0), & \text{if } r \leq \frac{2}{\sqrt{m}}, \\ \sqrt{m}\,\omega_1\left(f^{(1)}, rD_2(x_0)\right) D_2(x_0), & \text{if } r > \frac{2}{\sqrt{m}}. \end{cases}\end{aligned}$$

By Theorem 4.3 we obtain:

**Theorem 4.4.** *Let $f \in C_B\left(\mathbb{R}^N\right)$, whose Fréchet derivatives $f^{(j)}$, $1 \leq j \leq m$, $m \in \mathbb{N}$, exist, are bounded and continuous, $r > 0$, $x_0 \in \mathbb{R}^N$. Denote*

$$0 < \left(\widetilde{A}_n\left(\|\cdot - x_0\|^{m+1}\right)(x_0)\right)^{\frac{1}{(m+1)}} := D_{m+1}(x_0).$$

*Assume that*

$$w := \omega_1\left(f^{(m)}, rD_{m+1}(x_0)\right) > 0.$$

*Then*

$$\left|\widetilde{A}_n(f,x_0)-f(x_0)\right| \leq \left|\sum_{j=1}^{m}\frac{1}{j!}\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)(x_0)\right| + \frac{w}{rm!}\left[\frac{mr^2}{8}+\frac{r}{2}+\frac{1}{(m+1)}\right]D_{m+1}^{m}(x_0).$$

By Corollary 4.1 we get:

**Corollary 4.3.** *Denote*

$$D_2(x_0)=\left(\widetilde{A}_n\left(\|\cdot-x_0\|^2\right)(x_0)\right)^{\frac{1}{2}}$$

*which is positive, $x_0\in\mathbb{R}^N$. Assume that*

$$\omega_1\left(f^{(1)},\frac{1}{2}D_2(x_0)\right)>0.$$

*Then*

$$\left|\widetilde{A}_n(f,x_0)-f(x_0)\right|-\left|\widetilde{A}_n\left(f^{(1)}(x_0)(\cdot-x_0)\right)(x_0)\right| \leq (1.5625)\,\omega_1\left(f^{(1)},\frac{1}{2}D_2(x_0)\right)D_2(x_0).$$

By Proposition 4.1 we get:

**Proposition 4.2.** *Let $f\in C_B\left(\mathbb{R}^N\right)$, whose Fréchet derivative $f^{(1)}$ exists, is bounded and continuous, $r>0$, $x_0\in\mathbb{R}^N$. Denote*

$$D_2(x_0)=\left(\widetilde{A}_n\left(\|\cdot-x_0\|^2\right)(x_0)\right)^{\frac{1}{2}},$$

*which is positive. Then*

$$\left|\widetilde{A}_n(f,x_0)-f(x_0)\right|-\left|\widetilde{A}_n\left(f^{(1)}(x_0)(\cdot-x_0)\right)(x_0)\right| \leq \frac{1}{8r}(2+r)^2\,\omega_1\left(f',rD_2(x_0)\right)D_2(x_0).$$

By Corollary 4.2 we have:

**Corollary 4.4.** *Here all as in Proposition 4.2. Then*

$$\left|\widetilde{A}_n(f,x_0)-f(x_0)\right|-\left|\widetilde{A}_n\left(f^{(1)}(x_0)(\cdot-x_0)\right)(x_0)\right| \leq \begin{cases}\frac{1}{8r}(2+r)^2\,\omega_1\left(f^{(1)},rD_2(x_0)\right)D_2(x_0), & \text{if } r\leq 2,\\ \omega_1\left(f^{(1)},rD_2(x_0)\right)D_2(x_0), & \text{if } r>2.\end{cases}$$

We make

**Remark 4.1.** *We have that*

$$D_{m+1}(x_0)=\left(\widetilde{A}_n\left(\|\cdot-x_0\|^{m+1}\right)(x_0)\right)^{\frac{1}{(m+1)}} \overset{(3.1)}{\leq} \left(\frac{1}{n^\beta}+\frac{1}{n^{m+1}}\frac{\left(q+\frac{1}{q}\right)2^{m+2}(m+1)!}{\lambda^{m+1}(\ln A)^{m+1}}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right)^{\frac{1}{(m+1)}}\to 0,$$

*as $n \to +\infty$. And, furthermore we get*

$$D_2(x_0) = \left(\widetilde{A}_n\left(\|\cdot - x_0\|^2\right)(x_0)\right)^{\frac{1}{2}} \leq \sqrt{\left(\frac{1}{n^\beta} + \frac{1}{n^2}\frac{\left(q+\frac{1}{q}\right)16}{\lambda^2(\ln A)^2}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right)} \to 0,$$

*as $n \to +\infty$.*

We make

**Remark 4.2. On Theorem 4.4 :** *Assume that $f^{(j)}(x_0) = 0$, $j = 1, ..., m$. Then*

$$\left|\widetilde{A}_n(f, x_0) - f(x_0)\right|$$
$$\leq \frac{1}{rm!}\omega_1\left(f^{(m)}, r\left(\frac{1}{n^\beta} + \frac{1}{n^{m+1}}\frac{\left(q+\frac{1}{q}\right)2^{m+2}(m+1)!}{\lambda^{m+1}(\ln A)^{m+1}}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right)^{\frac{1}{(m+1)}}\right) \tag{4.3}$$
$$\times \left[\frac{mr^2}{8} + \frac{r}{2} + \frac{1}{(m+1)}\right]\left(\frac{1}{n^\beta} + \frac{1}{n^{m+1}}\frac{\left(q+\frac{1}{q}\right)2^{m+2}(m+1)!}{\lambda^{m+1}(\ln A)^{m+1}}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right)^{\frac{m}{(m+1)}} \to 0,$$

*as $n \to +\infty$. That is $\lim_{n\to+\infty} \widetilde{A}_n(f, x_0) = f(x_0)$.*

*Case $m = 1$ in (4.3) :*

*Assume $f^{(1)}(x_0) = 0$, then*

$$\left|\widetilde{A}_n(f, x_0) - f(x_0)\right|$$
$$\leq \frac{1}{r}\omega_1\left(f^{(1)}, r\left(\frac{1}{n^\beta} + \frac{1}{n^2}\frac{\left(q+\frac{1}{q}\right)16}{\lambda^2(\ln A)^2}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right)^{\frac{1}{2}}\right)$$
$$\times \left[\frac{r^2}{8} + \frac{r}{2} + \frac{1}{2}\right]\left(\frac{1}{n^\beta} + \frac{1}{n^2}\frac{\left(q+\frac{1}{q}\right)16}{\lambda^2(\ln A)^2}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right)^{\frac{1}{2}} \to 0, \quad as\ n \to +\infty;$$

*implying again $\lim_{n\to+\infty} \widetilde{A}_n(f, x_0) = f(x_0)$.*

We continue with general results and their applications.

**Theorem 4.5.** *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C_B(\mathbb{R}^N)$ into $B(\mathbb{R}^N)$, such that $L_n(1) = 1$. Assume $0 < L_n(\|\cdot - x_0\|_\infty)(x_0) < +\infty$. Then*

$$|L_n(f)(x_0) - f(x_0)| \leq 2\omega_1(f, L_n(\|\cdot - x_0\|_\infty)(x_0)) < +\infty. \tag{4.4}$$

*Given that $\lim_{n\to+\infty} L_n(\|\cdot - x_0\|_\infty)(x_0) = 0$ and $f$ is uniformly continuous, we get that $\lim_{n\to+\infty} L_n(f)(x_0) = f(x_0)$.*

*Proof.* We have that $\omega_1(f, h) < \infty$, $h > 0$. We got by (4.1) that

$$|f(\cdot) - f(x_0)| \leq \omega_1(f, h)\left[1 + \frac{\|\cdot - x_0\|_\infty}{h}\right], \quad \text{over } \mathbb{R}^N. \tag{4.5}$$

Hence we derive

$$\begin{aligned}\left|L_n(f)(x_0)-f(x_0)\right| &= \left|L_n(f(\cdot)-f(x_0))(x_0)\right| \\ &\leq L_n(\left|f(\cdot)-f(x_0)\right|)(x_0) \\ &\overset{(4.5)}{\leq} \omega_1(f,h)\left[1+\frac{L_n(\|\cdot-x_0\|_\infty)(x_0)}{h}\right]\end{aligned}$$

(choose $h := L_n(\|\cdot-x_0\|_\infty)(x_0) > 0$)

$$= 2\omega_1(f, L_n(\|\cdot-x_0\|_\infty)(x_0)).$$

□

Next we apply (4.4).

**Theorem 4.6.** *Let $n\in\mathbb{N} : n^{1-\beta}>2$, $0<\beta<1$, $x_0\in\mathbb{R}^N$. Then*

$$\left|\widetilde{K}_n(f)(x_0)-f(x_0)\right| \leq 2\omega_1\left(f,\widetilde{K}_n(\|\cdot-x_0\|_\infty)(x_0)\right) < +\infty,$$

*and*

$$\left|\widetilde{Q}_n(f)(x_0)-f(x_0)\right| \leq 2\omega_1\left(f,\widetilde{Q}_n(\|\cdot-x_0\|_\infty)(x_0)\right) < +\infty,$$

*and*

$$\left\{\begin{array}{c}\left\|\widetilde{K}_n(f)-f\right\|_\infty \\ \left\|\widetilde{Q}_n(f)-f\right\|_\infty\end{array}\right\}$$

$$\leq 2\omega_1\left(f,\left[\left(\frac{1}{n}+\frac{1}{n^\beta}\right)+\frac{1}{n}\left[\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}+\frac{\left(q+\frac{1}{q}\right)4}{\lambda\ln A}A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right]\right]\right) < +\infty.$$

*Given that $f\in C_U(\mathbb{R}^N)$ we derive that $\widetilde{K}_n(f)\to f$, $\widetilde{Q}_n(f)\to f$, as $n\to+\infty$, pointwise and uniformly.*

*Proof.* Along with (4.4) we use of Propositions 3.2, 3.3. □

We continue with the convexity case.

**Remark 4.3.** *Let $f\in C_B(\mathbb{R}^N)$ and $|f(\cdot)-f(x_0)|$ is convex over $\mathbb{R}^N$. Then, by (4.1), we have*

$$|f(\cdot)-f(x_0)| \leq \frac{\omega_1(f,h)}{h}\|\cdot-x_0\|_\infty,$$

*over $\mathbb{R}^N$, $x_0\in\mathbb{R}^N$.*

Acting as in Theorem 4.5 we derive:

**Theorem 4.7.** *Let $(L_n)_{n\in\mathbb{N}}$ positive linear operators from $C_B(\mathbb{R}^N)$ into $B(\mathbb{R}^N)$ : $L_n(1)=1$. Let $f\in C_B(\mathbb{R}^N)$ and the convex function $|f(\cdot)-f(x_0)|$ over $\mathbb{R}^N$, $x_0\in\mathbb{R}^N$. Assume that $0<L_n(\|\cdot-x_0\|_\infty)(x_0)<+\infty$. Then*

$$|L_n(f)(x_0)-f(x_0)| \leq \omega_1(f, L_n(\|\cdot-x_0\|_\infty)(x_0)). \tag{4.6}$$

*The last (4.6) is attained by $f(t)=\|t-x_0\|_\infty$, $\forall\ t\in\mathbb{R}^N$, $x_0\in\mathbb{R}^N$, that is sharp. Given that $f$ is uniformly continuous and $L_n(\|\cdot-x_0\|_\infty)(x_0)\to 0$, as $n\to\infty$, then $L_n(f)(x_0)\to f(x_0)$, as $n\to\infty$.*

Next we apply Theorem 4.7.

**Theorem 4.8.** *Let $f \in C_B\left(\mathbb{R}^N\right) : |f(\cdot) - f(x_0)|$ is convex, $x_0 \in \mathbb{R}^N$. Then*

$$\left\{ \begin{array}{l} \left|\widetilde{K}_n(f, x_0) - f(x_0)\right| \\ \left|\widetilde{Q}_n(f, x_0) - f(x_0)\right| \end{array} \right\}$$

$$\leq \quad \omega_1\left(f, \left[\left(\frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{1}{n}\left[\gamma A^{-\lambda\left(n^{1-\beta}-2\right)} + \frac{\left(q + \frac{1}{q}\right)4}{\lambda \ln A} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}\right]\right]\right) < +\infty.$$

*Given that $f \in C_U\left(\mathbb{R}^N\right)$, we obtain that $\widetilde{K}_n(f)(x_0) \to f(x_0)$, $\widetilde{Q}_n(f)(x_0) \to f(x_0)$, as $n \to +\infty$.*

*Proof.* By (4.6) and Propositions 3.2, 3.3. ☐

We make

**Remark 4.4.** *We consider $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$ and $\left(\mathbb{R}^N\right)^j$ denote the $j$-fold product space $\mathbb{R}^N \times ... \times \mathbb{R}^N$ endowed with the max-norm $\|x\|_{\left(\mathbb{R}^N\right)^j} := \max_{1 \leq k \leq j} \|x_k\|_\infty$, where $x := (x_1, ..., x_j) \in \left(\mathbb{R}^N\right)^j$. Then the space $L_j := L_j\left(\left(\mathbb{R}^N\right)^j; \mathbb{R}\right)$ of all real valued multilinear continuous functions $g : \left(\mathbb{R}^N\right)^j \to \mathbb{R}$ is a Banach space with norm*

$$\|g\| := \|g\|_{L_j} := \sup_{\left(\|x\|_{\left(\mathbb{R}^N\right)^j} = 1\right)} |g(x)| = \sup \frac{|g(x)|}{\|x_1\|_\infty ... \|x_j\|_\infty}.$$

*Let $x_0 \in \mathbb{R}^N$ be fixed, and $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous and bounded function whose Fréchet derivatives*

$$f^{(j)} : \mathbb{R}^N \to L_j = L_j\left(\left(\mathbb{R}^N\right)^j; \mathbb{R}\right)$$

*exist and are continuous and bounded for all $1 \leq j \leq m$, $m \in \mathbb{N}$. Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in \left(\mathbb{R}^N\right)^j = \underbrace{\mathbb{R}^N \times ... \times \mathbb{R}^N}_{-j-}$. Then, by Taylor's formula, see [26], p. 124 and [27], we get*

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_m(x, x_0), \quad \text{all } x \in \mathbb{R}^N,$$

*where*

$$R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!}\left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right)(x - x_0)^m du.$$

*Here we assume that $f^{(j)}(x_0) = 0$, $j = 1, ..., m$, then*

$$f(x) - f(x_0) = \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!}\left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right)(x - x_0)^m du.$$

*Consider*

$$w := \omega_1\left(f^{(m)}, h\right) = \sup_{\left\{\begin{array}{l} x, y \in \mathbb{R}^N \\ \|x - y\|_\infty \leq h \end{array}\right.} \left\|f^{(m)}(x) - f^{(m)}(y)\right\|, \quad h > 0.$$

*Notice that $w<+\infty$. We obtain that*

$$\begin{aligned}\left\|\left(f^{(m)}\left(x_0+u\left(x-x_0\right)\right)-f^{(m)}\left(x_0\right)\right)\left(x-x_0\right)^m\right\| &\leq \left\|f^{(m)}\left(x_0+u\left(x-x_0\right)\right)-f^{(m)}\left(x_0\right)\right\|\left\|x-x_0\right\|^m \\ &\leq w\left\|x-x_0\right\|^m\left\lceil\frac{u\left\|x-x_0\right\|}{h}\right\rceil,\end{aligned}$$

*by Lemma 7.1.1, p. 208, [4]. Therefore for all $x\in\mathbb{R}^N$ :*

$$\begin{aligned}\left|f(x)-f\left(x_0\right)\right| &\leq w\left\|x-x_0\right\|^m\int_0^1\left\lceil\frac{u\left\|x-x_0\right\|_\infty}{h}\right\rceil\frac{(1-u)^{m-1}}{(m-1)!}du \\ &= w\Phi_m\left(\left\|x-x_0\right\|_\infty\right),\end{aligned}$$

*by a change of variable, where*

$$\Phi_m(t):=\int_0^{|t|}\left\lceil\frac{s}{h}\right\rceil\frac{(|t|-s)^{m-1}}{(m-1)!}ds=\frac{1}{m!}\left(\sum_{j=0}^{\infty}(|t|-j\cdot h)_+^m\right), \tag{120}$$

*$t\in\mathbb{R}$; $(x)_+=\max\{x,0\}$. For a full list of properties of the spline function $\Phi_m$ see Remark 7.1.3 of [4], p. 210. By (7.1.18) of [4], p. 210, we have*

$$\Phi_m(x)\leq\left(\frac{|x|^{m+1}}{(m+1)!h}+\frac{|x|^m}{2m!}+\frac{h|x|^{m-1}}{8(m-1)!}\right),\quad\forall\,x\in\mathbb{R}.$$

*Consequently it holds ($x\in\mathbb{R}^N$)*

$$\left|f(x)-f\left(x_0\right)\right|\leq\omega_1\left(f^{(m)},h\right)\left(\frac{\left\|x-x_0\right\|_\infty^{m+1}}{(m+1)!h}+\frac{\left\|x-x_0\right\|_\infty^m}{2m!}+\frac{h\left\|x-x_0\right\|_\infty^{m-1}}{8(m-1)!}\right).$$

*So that*

$$\left|f(\cdot)-f\left(x_0\right)\right|\leq\omega_1\left(f^{(m)},h\right)\left(\frac{\left\|\cdot-x_0\right\|_\infty^{m+1}}{(m+1)!h}+\frac{\left\|\cdot-x_0\right\|_\infty^m}{2m!}+\frac{h\left\|\cdot-x_0\right\|_\infty^{m-1}}{8(m-1)!}\right), \tag{4.7}$$

*over $\mathbb{R}^N$. Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C_B\left(\mathbb{R}^N\right)$ into $B\left(\mathbb{R}^N\right)$ such that $L_n(1)=1$, $\forall\ n\in\mathbb{N}$. By $|f|\leq|f|$, iff $-|f|\leq f\leq|f|$, then $-L_n(|f|)\leq L_n(f)\leq L_n(|f|)$, and $|L_n(f)|\leq L_n(|f|)$. Hence*

$$\begin{aligned}&\left|L_n\left(f(\cdot)-f\left(x_0\right)\right)\left(x_0\right)\right|\leq L_n\left(\left|f(\cdot)-f\left(x_0\right)\right|\right)\left(x_0\right) \\ \overset{(4.7)}{\leq}\;&\omega_1\left(f^{(m)},h\right)\left[\frac{L_n\left(\left\|\cdot-x_0\right\|_\infty^{m+1}\right)\left(x_0\right)}{(m+1)!h}+\frac{L_n\left(\left\|\cdot-x_0\right\|_\infty^m\right)\left(x_0\right)}{2m!}+\frac{hL_n\left(\left\|\cdot-x_0\right\|_\infty^{m-1}\right)\left(x_0\right)}{8(m-1)!}\right] \\ =:\;&(\xi).\end{aligned} \tag{4.8}$$

We need the following Hölder's type inequality for positive linear operators.

**Theorem 4.9.** *Let $L$ be a positive linear operator from $C_B\left(\mathbb{R}^N\right)$ into $B\left(\mathbb{R}^N\right)$, and $f,g\in C_B\left(\mathbb{R}^N\right)$, furthermore let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Assume that $L\left(\left(|f(\cdot)|^p\right)\right)(s_*)$, $L\left(\left(|g(\cdot)|^q\right)\right)(s_*)>0$ for some $s_*\in\mathbb{R}^N$. Then*

$$L\left(|f(\cdot)g(\cdot)|\right)(s_*)\leq\left(L\left(\left(|f(\cdot)|^p\right)\right)(s_*)\right)^{\frac{1}{p}}\left(L\left(\left(|g(\cdot)|^q\right)\right)(s_*)\right)^{\frac{1}{q}}. \tag{4.9}$$

*Proof.* For $a, b \geq 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, the Young's inequality gives

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Then

$$\frac{|f(s)|}{\left(L\left(\left(|f(\cdot)|^p\right)\right)(s_*)\right)^{\frac{1}{p}}} \cdot \frac{|g(s)|}{\left(L\left(\left(|g(\cdot)|^q\right)\right)(s_*)\right)^{\frac{1}{q}}} \leq \frac{|f(s)|^p}{p\left(L\left(\left(|f(\cdot)|^p\right)\right)(s_*)\right)} + \frac{|g(s)|^q}{q\left(L\left(\left(|g(\cdot)|^q\right)\right)(s_*)\right)}, \ \forall\, s \in \mathbb{R}^N.$$

Therefore, we observe that

$$\frac{L\left(|f(\cdot)\, g(\cdot)|\right)(s_*)}{\left(L\left(\left(|f(\cdot)|^p\right)\right)(s_*)\right)^{\frac{1}{p}} \left(L\left(\left(|g(\cdot)|^q\right)\right)(s_*)\right)^{\frac{1}{q}}} \leq \frac{L\left(\left(|f(\cdot)|^p\right)\right)(s_*)}{p\left(L\left(\left(|f(\cdot)|^p\right)\right)(s_*)\right)} + \frac{L\left(\left(|g(\cdot)|^q\right)\right)(s_*)}{q\left(L\left(\left(|g(\cdot)|^q\right)\right)(s_*)\right)} = \frac{1}{p} + \frac{1}{q} = 1,$$

for $s_* \in \mathbb{R}^N$, which completes the proof. □

We continue with

**Remark 4.5.** *Let $g = 1$ and $L_n : L_n(1) = 1$, then by (4.9) we get:*

$$L_n\left(\|\cdot - x_0\|_\infty^m\right)(x_0) \leq \left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)},$$

*and*

$$L_n\left(\|\cdot - x_0\|_\infty^{(m-1)}\right)(x_0) \leq \left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\left(\frac{m-1}{m+1}\right)},$$

*where $m \in \mathbb{N}$ and $L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) > 0$. Continuing from (4.8):*

$$(\xi) \leq \omega_1\left(f^{(m)}, h\right)\left[\frac{L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)}{(m+1)!h}\right.$$
$$\left. + \frac{\left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{m}{m+1}}}{2m!} + \frac{h\left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{m-1}{m+1}}}{8(m-1)!}\right]$$

*(assume $h := \left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{1}{m+1}} > 0$)*

$$= \omega_1\left(f^{(m)}, \left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{1}{m+1}}\right)\left[\frac{h^m}{(m+1)!} + \frac{h^m}{2m!} + \frac{h^m}{8(m-1)!}\right]$$
$$= \omega_1\left(f^{(m)}, \left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{1}{m+1}}\right)$$
$$\times \left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{m}{m+1}}\left[\frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!}\right].$$

We have proved that

**Theorem 4.10.** *Let $f \in C_B(\mathbb{R}^N)$, $x_0 \in \mathbb{R}^N$, Fréchet derivatives $f^{(j)}$ continuous and bounded with $f^{(j)}(x_0) = 0$, $j = 1, ..., m$. Then*

$$\begin{aligned} & |L_n(f)(x_0) - f(x_0)| \\ \leq \; & \omega_1\left(f^{(m)}, \left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{1}{m+1}}\right)\left(L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{m}{m+1}} \\ \times \; & \left[\frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!}\right], \quad \forall\, n \in \mathbb{N}. \end{aligned}$$

*Here $L_n(1) = 1$, $L_n$ positive linear operators from $C_B(\mathbb{R}^N)$ into $B(\mathbb{R}^N)$, under the assumption $L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) > 0$. Let $L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) \to 0$, as $n \to +\infty$, then $\lim_{n\to+\infty} L_n(f)(x_0) = f(x_0)$.*

Now using Theorem 4.10 we get the next result for the operators $\widetilde{\Theta}_n$.

**Theorem 4.11.** *Let $f \in C_B(\mathbb{R}^N)$, $x_0 \in \mathbb{R}^N$, Fréchet derivatives $f^{(j)}$ continuous and bounded with $f^{(j)}(x_0) = 0$, $j = 1, ..., m$. Then*

$$\begin{aligned} & \left|\widetilde{\Theta}_n(f)(x_0) - f(x_0)\right| \\ \leq \; & \omega_1\left(f^{(m)}, \left(\widetilde{\Theta}_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{1}{m+1}}\right)\left(\widetilde{\Theta}_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{m}{m+1}} \\ \times \; & \left[\frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!}\right], \end{aligned}$$

*$\forall\, n \in \mathbb{N} : n^{1-\beta} > 2$, $0 < \beta < 1$. We get the pointwise approximation*

$$\lim_{n\to+\infty} \widetilde{\Theta}_n(f)(x_0) = f(x_0).$$

*Proof.* Here $\widetilde{\Theta}_n$ any of $\widetilde{A}_n$, $\widetilde{K}_n$, $\widetilde{Q}_n$. By Theorem 4.10, and Propositions 3.1-3.3 we get

$$\widetilde{\Theta}_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) \to 0,$$

as $n \to +\infty$. □

**Corollary 4.5.** *As established in Theorem 4.11, all assumptions remain valid here. Set*

$$\widetilde{\varphi}_1(n) := \frac{1}{n^\beta} + \frac{1}{n^{m+1}} \frac{\left(q + \frac{1}{q}\right) 2^{m+2}(m+1)!}{\lambda^{m+1}(\ln A)^{m+1}} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)}.$$

*Then*

$$\begin{aligned} \left|\widetilde{A}_n(f)(x_0) - f(x_0)\right| \leq \; & \omega_1\left(f^{(m)}, (\widetilde{\varphi}_1(n))^{\frac{1}{m+1}}\right) \\ \times \; & (\widetilde{\varphi}_1(n))^{\frac{m}{m+1}}\left[\frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!}\right], \end{aligned}$$

*$\forall\, n \in \mathbb{N} : n^{1-\beta} > 2$, $0 < \beta < 1$. We have that $\lim_{n\to+\infty} \widetilde{A}_n(f)(x_0) = f(x_0)$.*

*Proof.* It is clear from Theorem 4.11 and Proposition 3.1. □

We finish with

**Corollary 4.6.** *The assumptions and setting are the same as those in Theorem 4.11. Set*

$$\widetilde{\varphi}_2(n) := \left[ \left( \frac{1}{n} + \frac{1}{n^{\beta}} \right)^{m+1} + \frac{2^m}{n^{m+1}} \left( \frac{\left(q + \frac{1}{q}\right) 2^{m+2} (m+1)!}{\lambda^{m+1} (\ln A)^{m+1}} A^{-\frac{\lambda}{2}\left(n^{1-\beta}-3\right)} \right) \right].$$

*Then*

$$\left\{ \begin{array}{l} \left| \widetilde{K}_n(f)(x_0) - f(x_0) \right| \\ \left| \widetilde{Q}_n(f)(x_0) - f(x_0) \right| \end{array} \right\} \le \omega_1\left( f^{(m)}, (\widetilde{\varphi}_2(n))^{\frac{1}{m+1}} \right) \times (\widetilde{\varphi}_2(n))^{\frac{m}{m+1}} \left[ \frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!} \right],$$

$\forall\, n \in \mathbb{N} : n^{1-\beta} > 2$, $0 < \beta < 1$. *We get the pointwise approximation*

$$\lim_{n \to +\infty} \widetilde{K}_n(f)(x_0) = \lim_{n \to +\infty} \widetilde{Q}_n(f)(x_0) = f(x_0).$$

*Proof.* By Theorem 4.11 and Propositions 3.2, 3.3. □

## 5 Conclusion

This study has developed a rigorous approximation framework for multivariate generalized logistic neural network operators defined over the infinite domain. These operators have been interpreted as positive linear operators by employing symmetrized density functions constructed from parametrized and deformed generalized logistic activation functions.

We have established quantitative estimates for their convergence to the identity operator in both pointwise and uniform senses, using tools such as moduli of continuity and Fréchet derivatives. Additionally, strong connections to measure-theoretic formulations have been identified, which enrich the theoretical foundation of the analysis.

Refinements under convexity assumptions have been included. Moreover, three distinct families of operators, namely quasi interpolation, Kantorovich type, and quadrature type, have been investigated within a unified analytical framework. The derived bounds and asymptotic estimates confirm the reliability and effectiveness of the proposed constructions.

This contribution has reinforced the link between classical approximation theory and neural network operator approaches. Potential future research directions include extensions involving fractional operators, adaptively constructed densities, and applications in high-dimensional analysis and data-driven learning problems.

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