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Variational-Hemivariational Inequalities: A Brief Survey on Mathematical Theory and Numerical Analysis

Weimin Han 

Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

Abstract: Variational-hemivariational inequalities are an area full of interesting and challenging mathematical problems. The area can be viewed as a natural extension of that of variational inequalities. Variational-hemivariational inequalities are valuable for application problems from physical sciences and engineering that involve non-smooth and even set-valued relations, monotone or non-monotone, among physical quantities. In the recent years, there has been substantial growth of research interest in modeling, well-posedness analysis, development of numerical methods and numerical algorithms of variational-hemivariational inequalities. This survey paper is devoted to a brief account of well-posedness and numerical analysis results for variational-hemivariational inequalities. The theoretical results are presented for a family of abstract stationary variational-hemivariational inequalities and the main idea is explained for an accessible proof of existence and uniqueness. To better appreciate the distinguished feature of variational-hemivariational inequalities, for comparison, three mechanical problems are introduced leading to a variational equation, a variational inequality, and a variational-hemivariational inequality, respectively. The paper also comments on mixed variational-hemivariational inequalities, with examples from applications in fluid mechanics, and on results concerning the numerical solution of other types (nonstationary, history dependent) of variational-hemivariational inequalities.

Keywords: Variational inequality; hemivariational inequality; variational-hemivariational inequality; well-posedness; numerical solution; finite element method; discontinuous Galerkin method; virtual element method; convergence; error estimate; contact mechanics; Stokes hemivariational inequality; Navier-Stokes hemivariational inequality

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1 Introduction

It is generally agreed that the first variational inequality (VI) was formulated in A. Signorini's paper [90] in 1933 for a study of a frictionless contact problem between a linearized elastic body and a rigid foundation. The problem is commonly known as the Signorini problem. The existence and uniqueness of a solution to the Signorini problem was proved in early 1960s by G. Fishera (cf. [38]). This is considered as the beginning of the area of variational inequalities (VIs) ([4]). In mid 1960s to

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Email: weimin-han@uiowa.edu (W. Han)

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early 1970s, foundations of basic mathematical theory of VIs were established in a series of papers, cf. [13, 66, 78, 97]. The monograph [28] plays an important role in popularizing the area of VIs as it is shown in the book that many complicated application problems in mechanics and physics can be modeled and studied as VIs. Since there are no analytic solution formulas for VIs arising in applications, one relies on numerical methods to solve VIs. Early comprehensive references on numerical methods for solving VIs are [41, 42, 69]. Mechanics is a rich source of VIs, and there is a large number of references devoted to this topic, cf. [60, 100] on elasto-plasticity, [20, 29, 61, 67, 74, 91, 92, 94] on contact mechanics. Nowadays, active research persists in the area of VIs due to emerging new applications and the need of developing more efficient and effective numerical methods and algorithms to solve VIs (e.g., [17, 45, 71, 101, 119]).

VIs are featured by the presence of nonsmooth terms with a convex structure in their mathematical formulations. For applications, the nonsmooth convex structure often comes from a nonsmooth monotone relation among physical quantities of interest. For applications involving nonsmooth nonmonotone relations for physical quantities, hemivariational inequalities (HVIs) arise. P. D. Panagiotopoulos started the area of HVIs in early 1980s ([86]). VIs can be viewed as a special or degenerate case of HVIs in the sense that when the nonsmooth relations among physical quantities happen to be monotone, a HVI is reduced to a VI. We may consider the more general variational-hemivariational inequality (VHI) which contains nonsmooth terms of both kinds, those with a convex structure and those with a nonconvex structure. In the literature, the term HVI is often used to refer to a VHI also. In this paper, we use the terms VHI and HVI interchangeably.

Early comprehensive references in the area of VHIs include [82, 83, 87] on modeling, mathematical analysis and applications, and [68] for the finite element method to solve VHIs. In the last three decades, the area of VHIs has attracted the attention of ever more researchers, and recent comprehensive coverage of mathematical theory and applications can be found in [15, 16, 43, 44, 81, 96]. In these books and most math journal papers on VHIs, abstract surjectivity results on pseudomonotone operators are needed in proving the existence of a solution. Such an approach has its own merits. However, the requirement of the knowledge on abstract mathematical theory of pseudomonotone operators seems to be a hurdle to popularize the area of VHIs in the research communities of applied and computational mathematicians and engineers. An effort was made in [48, 49] in developing an alternative accessible approach for the mathematical theory of VHIs that does not rely on the abstract mathematical theory of pseudomonotone operators. We will describe the main idea of this alternative approach for studies of stationary VHIs in Subsection 4.1. We also note that the accessible approach is extended in [55, 56] for well-posedness of mixed VHIs. A comprehensive reference in these regards is the book [50].

Numerical methods are needed to solve VHIs. The finite element method and a variety of solution algorithms are discussed in [68] to solve HVIs. An optimal order error estimate is first presented in [57] for the linear finite element solutions of a VHI. This is followed by a series of papers on further analysis of the finite element method to solve VHIs, e.g., [47, 63–65]. The survey papers [62] and [52] provide accounts of recent advances on numerical analysis of VHIs. Besides the finite element method, other numerical methods such as the discontinuous Galerkin method and the virtual element method, have also been applied to solve VHIs, cf. discussions in Section 7. For the numerical solution of inequality problems (VIs, VHIs) of second-order, an optimal order error estimate can be achieved only for the linear element (linear finite element, linear virtual element) solutions (order one in the H^1 norm). Moreover, due to the limited regularity properties for solutions of inequality problems, low order elements are preferred to solve the inequalities.

The paper provides a summary account on the mathematical theory and numerical solution of VHIs and we focus on the stationary/time-independent problems. The main goal is to present the reader a relatively complete picture on the current status of the research on VHIs, especially regarding

the numerical analysis of VHIs. In Section 2, we review the notions of generalized subdifferentials and generalized subgradients, and their properties, as these are fundamental in the study of VHIs. In Section 3, we introduce three typical sample problems in mechanics in the order of increasing complexity. The first example is a variational equation (VE) for a standard boundary value problem. The other two examples are from contact mechanics, in the form of a VI and a VHI, respectively. It is hoped that through the presentation of the three examples, one can observe clearly what kind of features in a mechanical problem lead to a VI or a VHI. In Section 4, we consider an abstract stationary VHI. The first part of the section is to explain the main idea in the well-posedness analysis without the commonly used surjectivity results of pseudomonotone operators. The second part of the section summarizes main numerical analysis results on the abstract stationary VHI. The results presented in this section are general, and they can be applied to concrete VHIs, and to VIs and VEs which are special cases of VHIs. In Section 5, the theory presented in Section 4 is applied to the VHI contact problem introduced in Section 3, and we provide a well-posedness result and an optimal order error estimate for its numerical solution using the linear finite element method under certain solution regularity assumptions. In Section 6, we present sample results on Stokes and Navier-Stokes HVIs in fluid mechanics, as examples of mixed VHIs. In Section 7, we briefly comment on other numerical methods to solve VHIs and studies of time-dependent VHIs.

2 Generalized Directional Derivative, Subdifferential and Subgradient

In the study of nonsmooth problems, one fundamental issue is how to extend the concept of differentiability for functions that are not differentiable in the classical sense. For VIs, notions of subdifferential and subgradients serve such a purpose, and one can consult [30, 120] and many other excellent books on convex analysis for these notions. For VHIs, we need the notions of the generalized directional derivative and generalized subdifferential/subgradient for locally Lipschitz continuous functions introduced by F. H. Clarke ([23, 24]).

Let V be a Banach space, and let U be an open subset in V . Often, we can simply take $U = V$. Let $\Psi: U \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function. Given $u \in U$ and $v \in V$, the classical directional derivative $\Psi'(u; v)$ may not exist. The idea is to consider the ratio $(\Psi(w + \lambda v) - \Psi(w)) / \lambda$ for w close to u and $\lambda > 0$ close to 0. If the classical directional derivative $\Psi'(u; v)$ does not exist, then we cannot take the limit as $w \rightarrow u$ and $\lambda \downarrow 0$ on the ratio $(\Psi(w + \lambda v) - \Psi(w)) / \lambda$. Nevertheless, since Ψ is locally Lipschitz continuous at u , the upper limit of the ratio always exists as a real number. Thus, the generalized (or Clarke) directional derivative of Ψ at $u \in U$ in the direction $v \in V$ is defined by

$$\Psi^0(u; v) := \limsup_{w \rightarrow u, \lambda \downarrow 0} \frac{\Psi(w + \lambda v) - \Psi(w)}{\lambda}.$$

The next step is to define the generalized (or Clarke) subdifferential of Ψ at $u \in U$ by

$$\partial\Psi(u) := \{u^* \in V^* \mid \Psi^0(u; v) \geq \langle u^*, v \rangle \ \forall v \in V\}. \quad (2.1)$$

It can be shown that $\partial\Psi(u)$ is nonempty, convex, and weakly* compact in V^* . Any element in the set $\partial\Psi(u)$ is called a generalized (or Clarke) subgradient of Ψ at $u \in U$.

Furthermore, it can be proved that in case $\Psi: U \rightarrow \mathbb{R}$ happens to be convex, then the generalized subdifferential $\partial\Psi(u)$ at any $u \in U$ coincides with the convex subdifferential $\partial\Psi(u)$. Because of this property, it is perfectly reasonable to use the same symbol ∂ for both the generalized subdifferential of locally Lipschitz continuous functions and the convex subdifferential of convex functions.

Properties of the generalized directional derivative and the generalized subdifferential can be found in several books, e.g., [24, 26] or [81, Section 3.2]. In the following, we mention some basic properties to help those readers without prior exposures to the notions of the generalized directional derivative and the generalized subdifferential for a better understanding.

The generalized directional derivative is non-negatively homogeneous and sub-additive with respect to the direction variable:

$$\begin{aligned}\Psi^0(u; \lambda v) &= \lambda \Psi^0(u; v) \quad \forall \lambda \geq 0, u \in U, v \in V, \\ \Psi^0(u; v_1 + v_2) &\leq \Psi^0(u; v_1) + \Psi^0(u; v_2) \quad \forall u \in U, v_1, v_2 \in V.\end{aligned}$$

The generalized subdifferential is defined by the formula (2.1) through the use of the generalized directional derivative. Conversely, knowing the generalized subdifferential, we can compute the generalized directional derivative by the formula

$$\Psi^0(u; v) = \max \{ \langle u^*, v \rangle \mid u^* \in \partial \Psi(u) \} \quad \forall u \in U, v \in V. \quad (2.2)$$

In the study of VHIs and their numerical approximations, limiting properties of the generalized subdifferential and the generalized directional derivative are useful and we list two such properties below:

If $u_n \rightarrow u$ in V , $u_n, u \in U$, and $v_n \rightarrow v$ in V , then

$$\limsup_{n \rightarrow \infty} \Psi^0(u_n; v_n) \leq \Psi^0(u; v).$$

If $u_n \rightarrow u$ in V , $u_n, u \in U$, $u_n^* \in \partial \Psi(u_n)$, and $u_n^* \rightarrow u^*$ weakly* in V^* , then $u^* \in \partial \Psi(u)$.

Next, let $\Psi, \Psi_1, \Psi_2: U \rightarrow \mathbb{R}$ be locally Lipschitz functions. Then, we have the scalar multiplication rule

$$\partial(\lambda \Psi)(u) = \lambda \partial \Psi(u) \quad \forall \lambda \in \mathbb{R}, u \in U,$$

and the summation rule

$$\partial(\Psi_1 + \Psi_2)(u) \subset \partial \Psi_1(u) + \partial \Psi_2(u) \quad \forall u \in U, \quad (2.3)$$

or equivalently,

$$(\Psi_1 + \Psi_2)^0(u; v) \leq \Psi_1^0(u; v) + \Psi_2^0(u; v) \quad \forall u \in U, v \in V. \quad (2.4)$$

Moreover, (2.3) and (2.4) hold with equalities if Ψ_1 and Ψ_2 are regular at u . The regularity of the Lipschitz continuous function $\Psi: U \rightarrow \mathbb{R}$ at $u \in U$ means that the directional derivative $\Psi'(u; v)$ exists and

$$\Psi'(u; v) = \Psi^0(u; v) \quad \forall v \in V. \quad (2.5)$$

It is known that a function is regular at any point where the function is continuously differentiable. Also, a convex function is regular in the interior of its effective domain.

In the study of VHIs, we will assume that there exists a constant $\alpha_\Psi \geq 0$ such that

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in U. \quad (2.6)$$

This condition characterizes the degree of non-convexity of the functional Ψ : the smaller the constant $\alpha_\Psi \geq 0$, the weaker the non-convexity of Ψ . For a convex functional Ψ , (2.6) holds with $\alpha_\Psi = 0$. The condition (2.6) is sometimes expressed equivalently as a condition on the generalized subdifferential:

$$\langle v_1^* - v_2^*, v_1 - v_2 \rangle \geq -\alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall v_i \in U, v_i^* \in \partial \Psi(v_i), i = 1, 2. \quad (2.7)$$

In the literature, (2.7) is usually called a relaxed monotonicity condition. The inequality (2.7) with $\alpha_\Psi = 0$ is the monotonicity of $\partial\Psi$ for a convex functional Ψ . A short-hand expression of the condition (2.7) is

$$\langle \partial\Psi(v_1) - \partial\Psi(v_2), v_1 - v_2 \rangle \geq -\alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in U. \quad (2.8)$$

It can be proved (e.g., [50, p. 26]) that the condition (2.6) holds if and only if the functional $v \mapsto \Psi(v) + (\alpha_\Psi/2) \|v\|_V^2$ is convex on U . This result provides a simple way to verify the condition (2.6).

In virtually all the applications in mechanics, the locally Lipschitz continuous functional Ψ takes the form of an integral of a locally Lipschitz continuous function ψ of a real variable or of several real variables. For a locally Lipschitz continuous function defined over a finite dimensional set, there is a useful formula to compute the generalized subdifferential (cf. [25, Theorem 10.7], [81, Prop. 3.34]).

Proposition 2.1. *Assume $\Omega \subset \mathbb{R}^d$ is open, $\psi: \Omega \rightarrow \mathbb{R}$ is locally Lipschitz continuous near $\mathbf{x} \in \Omega$, $D \subset \mathbb{R}^d$ with $|D| = 0$, and $D_\psi \subset \mathbb{R}^d$ with $|D_\psi| = 0$ such that ψ is Fréchet differentiable on $\Omega \setminus D_\psi$. Then,*

$$\partial\psi(\mathbf{x}) = \text{conv} \{ \lim \psi'(\mathbf{x}_k) \mid \mathbf{x}_k \rightarrow \mathbf{x}, \mathbf{x}_k \notin D \cup D_\psi \}.$$

Next, we show some examples to compute the generalized subdifferential of locally Lipschitz continuous functions by applying Proposition 2.1 to conclude this section.

For the function $\psi_1(x) = -|x|$, $x \in \mathbb{R}$, its generalized subdifferential coincides with its classical derivative -1 for $x > 0$, and 1 for $x < 0$. At $x = 0$, $\partial\psi_1(0)$ equals the convex hull of the two derivative limiting points -1 and 1 . Thus,

$$\partial\psi_1(x) = \begin{cases} 1 & \text{if } x < 0, \\ [-1, 1] & \text{if } x = 0, \\ -1 & \text{if } x > 0. \end{cases}$$

For the generalized directional derivative $\psi_1(x; y)$, $x, y \in \mathbb{R}$, we can use the property (2.2) to find that

$$\psi_1^0(x; y) = \begin{cases} y & \text{if } x < 0, \\ |y| & \text{if } x = 0, \\ -y & \text{if } x > 0. \end{cases}$$

Similarly, for $\psi_2(x) = |x|$, $x \in \mathbb{R}$, its generalized subdifferential is

$$\partial\psi_2(x) = \begin{cases} -1 & \text{if } x < 0, \\ [-1, 1] & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Note that ψ_2 is a convex function, and $\partial\psi_2$ is also the convex subdifferential of ψ_2 . Moreover, the generalized directional derivative is, for any $y \in \mathbb{R}$,

$$\psi_2^0(x; y) = \begin{cases} -y & \text{if } x < 0, \\ |y| & \text{if } x = 0, \\ y & \text{if } x > 0. \end{cases}$$

As a more complicated example, consider

$$\psi_3(x) = \begin{cases} \sin x & \text{if } x < 0, \\ -x^2 & \text{if } 0 \leq x \leq 1, \\ x^3 - 3x^2 + 3x - 2 & \text{if } x > 1. \end{cases}$$

For its generalized subdifferential, we can readily write down the formula

$$\partial\psi_3(x) = \begin{cases} \cos x & \text{if } x < 0, \\ [0, 1] & \text{if } x = 0, \\ -2x & \text{if } 0 < x < 1, \\ [-2, 0] & \text{if } x = 1, \\ 3(x-1)^2 & \text{if } x > 1. \end{cases}$$

For the directional derivative at points where the classical derivative does not exist, we find that

$$\begin{aligned} \psi_3^0(0; y) &= \max\{0, y\}, \\ \psi_3^0(1; y) &= \max\{0, -2y\} \end{aligned}$$

for any $y \in \mathbb{R}$.

3 Three Representative Problems in Linearized Elasticity

We will see an example of a VHI in Subsection 3.4, for a contact problem. For comparison, under similar mechanical settings with simpler boundary conditions, we will provide an example of a variational equation in Subsection 3.2, and an example of a VI in Subsection 3.3. We will first introduce the notation in Subsection 3.1.

3.1 Notation

Consider mathematical models describing the equilibrium state of an elastic body subject to the action of external forces and constraints on the boundary. Let the reference configuration of the body be the closure of an open, bounded and connected set Ω in $\mathbb{R}^d$. For applications, the dimension $d = 2$ or 3 . Assume Ω has a Lipschitz boundary $\Gamma = \partial\Omega$. Then, the unit outward normal vector $\boldsymbol{\nu}$ exists a.e. on Γ . We use boldface letters for vectors and tensors. A typical point in $\mathbb{R}^d$ is denoted by $\mathbf{x} = (x_i)$. The range of indices i, j, k, l is between 1 and d . We adopt the summation convention over a repeated index. For convenience, for a subset Δ of Ω or that of Γ , and a function g defined on Δ , we use $I_\Delta(g)$ to denote the integral of the function g over Δ .

We denote by $\mathbb{S}^d$ the space of real symmetric matrices of order d . Over $\mathbb{R}^d$ and $\mathbb{S}^d$, we use the canonical inner products and norms:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i, \quad |\mathbf{v}| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \quad \forall \mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^d, \\ \boldsymbol{\sigma} : \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2} \quad \forall \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d. \end{aligned}$$

The primary unknown of the mechanical problems in this section is the displacement of the elastic body, $\mathbf{u}: \bar{\Omega} \rightarrow \mathbb{R}^d$. The linearized strain tensor is

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

which is an $\mathbb{S}^d$ -valued field in Ω . In componentwise form,

$$\varepsilon_{ij}(\mathbf{u}) = (\boldsymbol{\varepsilon}(\mathbf{u}))_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad 1 \leq i, j \leq d,$$

where $u_{i,j} = \partial u_i / \partial x_j$. In mechanical problems, another important quantity is the stress tensor $\boldsymbol{\sigma}: \Omega \rightarrow \mathbb{S}^d$, which is also an $\mathbb{S}^d$ -valued field in Ω .

We will use Sobolev and Lebesgue spaces on Ω , Γ , or their subsets, such as $L^2(\Omega; \mathbb{R}^d)$, $L^2(\Gamma_N; \mathbb{R}^d)$, $L^2(\Gamma_C; \mathbb{R}^d)$, $H^1(\Omega; \mathbb{R}^d)$, and $H^1(\Omega; \mathbb{S}^d)$, endowed with their canonical inner products and associated norms. Here, Γ_N and Γ_C are measurable subsets of Γ . For a function $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$ we write $\mathbf{v}$ for its trace $\gamma \mathbf{v} \in L^2(\Gamma; \mathbb{R}^d)$ on Γ . A standard reference on Sobolev spaces is [1]. One may also consult many other books on Sobolev spaces or PDEs, e.g., [14, 31].

The space for the stress field is

$$\mathbb{Q} := L^2(\Omega; \mathbb{S}^d) = \{ \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega), \ 1 \leq i, j \leq d \}. \quad (3.1)$$

This is a real Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathbb{Q}} = \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\tau} \, dx, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{Q}.$$

The corresponding norm is denoted by $\| \cdot \|_{\mathbb{Q}}$.

Let Γ_D be a non-trivial measurable subset of Γ . We will specify a homogeneous displacement boundary condition on Γ_D , and seek the unknown displacement field in the space

$$\mathbf{V} := \left\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\} \quad (3.2)$$

or its subspace. Since $|\Gamma_D| > 0$, Korn's inequality asserts that there is a constant $c > 0$, depending on Ω and Γ_D , such that

$$\| \mathbf{v} \|_{H^1(\Omega; \mathbb{R}^d)} \leq c \| \boldsymbol{\varepsilon}(\mathbf{v}) \|_{\mathbb{Q}} \quad \forall \mathbf{v} \in \mathbf{V}. \quad (3.3)$$

A proof of the Korn inequality can be found in numerous publications, e.g. [84, p. 79]. As a result, $\mathbf{V}$ is a real Hilbert space under the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}} = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathbb{Q}}. \quad (3.4)$$

The induced norm is

$$\| \mathbf{v} \|_{\mathbf{V}} = \| \boldsymbol{\varepsilon}(\mathbf{v}) \|_{\mathbb{Q}}. \quad (3.5)$$

It follows from Korn's inequality (3.3) that $\| \cdot \|_{H^1(\Omega; \mathbb{R}^d)}$ and $\| \cdot \|_{\mathbf{V}}$ are equivalent norms on $\mathbf{V}$. We will use $\| \cdot \|_{\mathbf{V}}$ as the norm on $\mathbf{V}$ or its subspace. Denote by $\mathbf{V}^*$ the dual of the space $\mathbf{V}$ and by $\langle \cdot, \cdot \rangle$ the corresponding duality pairing. For any element $\mathbf{v} \in \mathbf{V}$, define its normal and tangential components on Γ by $v_{\nu} = \mathbf{v} \cdot \boldsymbol{\nu}$ and $\mathbf{v}_{\tau} = \mathbf{v} - v_{\nu} \boldsymbol{\nu}$, respectively. Similarly, for a function $\boldsymbol{\sigma} : \bar{\Omega} \rightarrow \mathbb{S}^d$, its normal and tangential components on Γ are $\sigma_{\nu} = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_{\tau} = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_{\nu} \boldsymbol{\nu}$, respectively.

For a differentiable field $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$, its divergence is a vector-valued function $\operatorname{div} \boldsymbol{\sigma} : \bar{\Omega} \rightarrow \mathbb{R}^d$ with components

$$(\operatorname{div} \boldsymbol{\sigma})_i = \sigma_{ij,j}, \quad 1 \leq i \leq d,$$

where the summation convention over the repeated index j is applied. For $\boldsymbol{\sigma} \in H^1(\Omega; \mathbb{S}^d)$ and $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$, we have Green's formula (integration-by-parts formula)

$$\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \operatorname{div} \boldsymbol{\sigma} \cdot \mathbf{v} \, dx = \int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \mathbf{v} \, ds,$$

which is applied repeatedly in derivations of weak formulations.

3.2 A variational equation problem in linearized elasticity

The first mechanical problem leads to a variational equation. In this subsection only, the boundary $\Gamma = \Gamma_D \cup \Gamma_N$ is split into two parts Γ_D and Γ_N , with $|\Gamma_D| > 0$. If $\Gamma_N = \emptyset$, then $\Gamma_D = \Gamma$ is the entire boundary. The pointwise formulation of the problem is

$$-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f}_0 \quad \text{in } \Omega, \quad (3.6)$$

$$\boldsymbol{\sigma} = \mathcal{E} \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \quad (3.7)$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \quad \text{in } \Omega, \quad (3.8)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (3.9)$$

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_N. \quad (3.10)$$

We comment that (3.6) is the equilibrium equation, $\mathbf{f}_0$ being the density function of the body force; (3.7) is the elastic constitutive law, $\mathcal{E}$ being the elasticity tensor; (3.8) is the defining relation for the linearized strain tensor; (3.9) represents the homogeneous boundary condition on Γ_D ; and (3.10) describes the traction boundary condition on Γ_N , $\mathbf{f}_2$ being the density function of the traction force.

In the general case, the elasticity operator $\mathcal{E}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ in the constitutive law (3.7) is allowed to depend on the spatial location. For homogeneous materials, $\mathcal{E}: \mathbb{S}^d \rightarrow \mathbb{S}^d$ does not depend on the spatial variable. We assume the following properties:

$$\left\{ \begin{array}{l} \text{(a) There exists a constant } L_{\mathcal{E}} > 0 \text{ such that a.e. in } \Omega, \\ \quad |\mathcal{E}(\cdot, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\cdot, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{E}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d; \\ \text{(b) there exists a constant } m_{\mathcal{E}} > 0 \text{ such that a.e. in } \Omega, \\ \quad (\mathcal{E}(\cdot, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\cdot, \boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{E}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d; \\ \text{(c) } \mathcal{E}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(d) } \mathcal{E}(\cdot, \mathbf{0}) = \mathbf{0} \text{ a.e. in } \Omega. \end{array} \right. \quad (3.11)$$

For the force densities, we assume

$$\mathbf{f}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \mathbf{f}_2 \in L^2(\Gamma_N; \mathbb{R}^d), \quad (3.12)$$

and define an element $\mathbf{f} \in V^*$ by

$$\langle \mathbf{f}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} \, dx + \int_{\Gamma_N} \mathbf{f}_2 \cdot \mathbf{v} \, ds \quad \forall \mathbf{v} \in V. \quad (3.13)$$

Through a standard procedure, one can derive the following weak formulation of the problem defined by (3.6)–(3.10); for such a derivation, cf. e.g., [50, Section 4.2].

Problem 3.1. Find a displacement field $\mathbf{u} \in V$ such that

$$(\mathcal{E}(\boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathbb{Q}} = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V, \quad (3.14)$$

where $\mathbf{f} \in V^*$ is defined by (3.13).

Well-posedness of Problem 3.1 can be shown through an application of the well-known Lax-Milgram lemma; for detail, one may consult [50, Theorem 4.1].

3.3 A variational inequality in contact mechanics

In the study of contact problems, we assume the boundary of Γ of the domain Ω is decomposed into three non-overlapping measurable parts: $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$, with $|\Gamma_D| > 0$ and $|\Gamma_C| > 0$. We will specify a displacement boundary condition on Γ_D , a traction boundary condition on Γ_N , and contact boundary conditions on Γ_C . A variety of mathematical models of contact problems can be found in many publications, cf. e.g., the comprehensive references [29, 61, 74, 81, 92, 94, 109].

For the sample contact problem considered in this subsection, the pointwise relations (3.6)–(3.10) are supplemented by a bilateral contact condition with a Tresca's friction law on Γ_C :

$$\begin{aligned} u_\nu &= 0, \\ |\boldsymbol{\sigma}_\tau| &\leq f_b \quad \text{and} \\ |\boldsymbol{\sigma}_\tau| < f_b &\Rightarrow \mathbf{u}_\tau = \mathbf{0}, \\ |\boldsymbol{\sigma}_\tau| = f_b &\Rightarrow \mathbf{u}_\tau = -\lambda \boldsymbol{\sigma}_\tau \text{ for some } \lambda \geq 0. \end{aligned} \quad (3.15)$$

Here $f_b \geq 0$ is a constant bound of the magnitude of the friction force.

A study of this contact problem can be found in several references, e.g., [50, Subsection 4.4.1]. The tangential contact conditions in (3.15) can be equivalently expressed as

$$|\boldsymbol{\sigma}_\tau| \leq f_b \quad \text{and} \quad \boldsymbol{\sigma}_\tau \cdot \mathbf{u}_\tau + f_b |\mathbf{u}_\tau| = 0,$$

or as

$$|\boldsymbol{\sigma}_\tau| \leq f_b, \quad -\boldsymbol{\sigma}_\tau = f_b \frac{\mathbf{u}_\tau}{|\mathbf{u}_\tau|} \text{ if } \mathbf{u}_\tau \neq \mathbf{0},$$

or as

$$-\boldsymbol{\sigma}_\tau \in \partial\phi(\mathbf{u}_\tau), \quad \phi(\mathbf{z}) = f_b |\mathbf{z}| \text{ for } \mathbf{z} \in \mathbb{R}^d.$$

Here, ϕ is a convex function and $\partial\phi$ stands for the (convex) subdifferential of ϕ .

Define a subspace of the space $\mathbf{V}$:

$$\mathbf{V}_1 = \{\mathbf{v} \in \mathbf{V} \mid v_\nu = 0 \text{ on } \Gamma_C\} \quad (3.16)$$

and use the norm $\|\cdot\|_{\mathbf{V}}$ over the subspace $\mathbf{V}_1$. The weak formulation of the contact problem is the following VI (cf. [50, Problem 4.6]).

Problem 3.2. Find $\mathbf{u} \in \mathbf{V}_1$ such that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}))_{\mathbb{Q}} + I_{\Gamma_C}(f_b |\mathbf{v}_\tau|) - I_{\Gamma_C}(f_b |\mathbf{u}_\tau|) \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle \quad \forall \mathbf{v} \in \mathbf{V}_1, \quad (3.17)$$

where $\mathbf{f} \in \mathbf{V}^*$ is defined by (3.13).

Well-posedness of this problem is the content of [50, Theorem 4.7]. See also Section 5.

3.4 A variational-hemivariational inequality in contact mechanics

In this subsection, we consider another contact problem whose mathematical model is a VHI. As in the previous subsection, $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$, with $|\Gamma_D| > 0$ and $|\Gamma_C| > 0$. The pointwise relations (3.6)–(3.10) are supplemented by the normal compliance contact condition ([81, Section 6.3]) with Tresca's friction law on the contact boundary Γ_C :

$$\begin{aligned} -\sigma_\nu &\in \partial\psi_\nu(u_\nu), \\ |\boldsymbol{\sigma}_\tau| &\leq f_b \quad \text{and} \\ |\boldsymbol{\sigma}_\tau| < f_b &\Rightarrow \mathbf{u}_\tau = \mathbf{0}, \\ |\boldsymbol{\sigma}_\tau| = f_b &\Rightarrow \mathbf{u}_\tau = -\lambda \boldsymbol{\sigma}_\tau \text{ for some } \lambda \geq 0. \end{aligned} \quad (3.18)$$

Here, the function $\psi_\nu: \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz continuous and is not necessarily convex, $f_b \geq 0$ is a constant bound of the magnitude of the friction force. In particular, when $f_b = 0$, the last two relations in (3.18) degenerate to the frictionless condition

$$-\sigma_\tau = \mathbf{0} \quad \text{on } \Gamma_C.$$

We assume the following properties on the function $\psi_\nu: \mathbb{R} \rightarrow \mathbb{R}$:

$$\left\{ \begin{array}{l} \text{(a) } \psi_\nu(\cdot) \text{ is locally Lipschitz on } \mathbb{R}; \\ \text{(b) there exist constants } \bar{c}_0, \bar{c}_1 \geq 0 \text{ such that} \\ \quad |\partial\psi_\nu(z)| \leq \bar{c}_0 + \bar{c}_1 |z| \quad \forall z \in \mathbb{R}; \\ \text{(c) there exists a constant } \alpha_{\psi_\nu} \geq 0 \text{ such that} \\ \quad \psi_\nu^0(z_1; z_2 - z_1) + \psi_\nu^0(z_2; z_1 - z_2) \leq \alpha_{\psi_\nu} |z_1 - z_2|^2 \quad \forall z_1, z_2 \in \mathbb{R}. \end{array} \right. \quad (3.19)$$

The weak formulation of the contact problem of (3.6)–(3.10) and (3.18) is the following VHI (cf. [57], [50, Problem 4.20]).

Problem 3.3. Find a displacement field $\mathbf{u} \in \mathbf{V}$ such that

$$\begin{aligned} & (\mathcal{E}(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}))_{\mathbb{Q}} + I_{\Gamma_C}(f_b|\mathbf{v}_\tau|) - I_{\Gamma_C}(f_b|\mathbf{u}_\tau|) + I_{\Gamma_C}(\psi_\nu^0(u_\nu; v_\nu - u_\nu)) \\ & \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \quad (3.20)$$

Well-posedness of Problem 3.3 is stated in Theorem 5.1. The more general case of a friction bound $f_b = f_b(u_\nu)$ is considered in [57], see also [50, Example 5.51].

4 An Abstract Stationary Variational-Hemivariational Inequality

Problem 3.3 can be studied within the framework of an abstract stationary VHI discussed in this section. Problem 3.1 and Problem 3.2 can be studied as special cases of a VI and a VE of the abstract stationary VHI. Denote by Δ the physical domain or its sub-domain, or its boundary or part of the boundary, and denote by I_Δ the integration over Δ ,

$$I_\Delta(v) = \int_\Delta v \, dx \text{ if } \Delta \subset \Omega, \quad I_\Delta(v) = \int_\Delta v \, ds \text{ if } \Delta \subset \Gamma.$$

For a positive integer m , we let

$$V_\psi = L^2(\Delta; \mathbb{R}^m). \quad (4.1)$$

For application in the study of Problem 3.3, we take $m = 1$; for some other contact problems (e.g., Problem 4.18 in [50]), we take $m = d$.

We first introduce the following assumptions on the data for the abstract VHI.

$H(V)$ V is a real Hilbert space.

$H(K)$ K is a non-empty, closed and convex set in V .

$H(A)$ $A: V \rightarrow V^*$ is L_A -Lipschitz continuous and m_A -strongly monotone.

$H(\Phi)$ $\Phi: V \rightarrow \mathbb{R}$ is convex and continuous on V .

$H(\psi)$ $\gamma_\psi \in \mathcal{L}(V; V_\psi)$; $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ is locally Lipschitz continuous and for some non-negative constants c_ψ and α_ψ ,

$$|\partial\psi(z)|_{\mathbb{R}^m} \leq c_\psi (1 + |z|_{\mathbb{R}^m}) \quad \forall z \in \mathbb{R}^m, \quad (4.2)$$

$$\psi^0(z_1; z_2 - z_1) + \psi^0(z_2; z_1 - z_2) \leq \alpha_\psi |z_1 - z_2|_{\mathbb{R}^m}^2 \quad \forall z_1, z_2 \in \mathbb{R}^m. \quad (4.3)$$

$H(f)$ $f \in V^*$.

Note that an operator $A: V \rightarrow V^*$ is said to be L_A -Lipschitz continuous if

$$\|Av_1 - Av_2\|_{V^*} \leq L_A \|v_1 - v_2\|_V \quad \forall v_1, v_2 \in V, \quad (4.4)$$

and it is said to be m_A -strongly monotone if

$$\langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in V. \quad (4.5)$$

A consequence of the assumption $H(\Phi)$ is that $\Phi(\cdot)$ is bounded below by a function that grows at most linearly, i.e., for some constants c_3 and c_4 , not necessarily positive,

$$\Phi(v) \geq c_3 + c_4 \|v\|_V \quad \forall v \in V, \quad (4.6)$$

cf. e.g., [5, Lemma 11.3.5].

Generally, we can consider the situation where $\psi = \psi(\mathbf{x}, z)$ is a function defined for $\mathbf{x} \in \Delta$ and $z \in \mathbb{R}^m$. To simplify the exposition, we will only consider the case where $\psi = \psi(z)$ does not depend on $\mathbf{x} \in \Delta$.

The condition (4.3) is equivalent to the following inequality:

$$\langle v_1^* - v_2^*, v_1 - v_2 \rangle \geq -\alpha_\Psi |v_1 - v_2|_{\mathbb{R}^m}^2 \quad \forall v_i \in \mathbb{R}^m, v_i^* \in \partial\psi(v_i), i = 1, 2. \quad (4.7)$$

It can be shown (e.g., Lemma 4.2 in [52]) that under the assumption $H(\psi)$,

$$|I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v))| \leq c(1 + \|u\|_V) \|\gamma_\psi v\|_{V_\psi} \quad \forall u, v \in V. \quad (4.8)$$

The abstract VHI is the following.

Problem 4.1. Find $u \in K$ such that

$$\langle Au, v - u \rangle + \Phi(v) - \Phi(u) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \geq \langle f, v - u \rangle \quad \forall v \in K. \quad (4.9)$$

In [50], this problem is called a VHI of rank $(1, 1)$ to reflect the fact that in the variational-hemivariational inequality (4.9), the convex function Φ depends on one argument and the locally Lipschitz continuous function ψ depends on one argument. In the general case $K \neq V$, Problem 4.1 can be viewed as a constrained VHI of rank $(1, 1)$.

When $K = V$ is the entire space, Problem 4.1 becomes an unconstrained VHI of rank $(1, 1)$: Find $u \in V$ such that

$$\langle Au, v - u \rangle + \Phi(v) - \Phi(u) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \geq \langle f, v - u \rangle \quad \forall v \in V. \quad (4.10)$$

4.1 Well-posedness of the abstract stationary variational-hemivariational inequality

As an intermediate step in the well-posedness study of Problem 4.1, we analyze an auxiliary VHI.

Problem 4.2. Find $u \in K$ such that

$$\langle Au, v - u \rangle + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall v \in K. \quad (4.11)$$

In the analysis of Problem 4.2, we assume $H(V)$, $H(K)$, $H(A)$, $H(\Phi)$, $H(f)$, and $H(\Psi)$ $\Psi: V \rightarrow \mathbb{R}$ is locally Lipschitz continuous, and for some constant $\alpha_\Psi \geq 0$,

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in V.$$

In the majority of the references on well-posedness analysis of VHIs, the operator $A: V \rightarrow V^*$ is assumed pseudomonotone, coercive, and strongly monotone, and abstract surjectivity results for pseudomonotone operators (e.g., [83, Theorem 2.12], [27, Theorem 1.3.70]) are applied (cf. [81, 96]). While such an approach carries its own merit, it is desirable to have a more accessible approach for applied and computational mathematicians, and for engineers. One accessible approach is developed in [48, 49], and it does not require knowledge on pseudomonotone operators or abstract analysis. We provide a summarizing account of this approach below. For details, the reader can consult the book [50].

In the first step of the accessible approach, we assume additionally that $A: V \rightarrow V^*$ is a potential operator, i.e., it is the Gâteaux derivative of a functional $F_A: V \rightarrow \mathbb{R}$. This allows us to define a related optimization problem.

Problem 4.3. Find $u \in K$ such that

$$E(u) = \inf \{E(v) \mid v \in K\}$$

where the energy functional

$$E(v) = F_A(v) + \Phi(v) + \Psi(v) - \langle f, v \rangle, \quad v \in V.$$

Problem 4.2 is analyzed through Problem 4.3.

Theorem 4.1. Assume $H(V)$, $H(K)$, $H(A)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, and $\alpha_\Psi < m_A$. Assume additionally that A is a potential operator with the potential F_A . Then, Problem 4.3 has a unique solution $u \in K$, which is also the unique solution of Problem 4.2.

In the second step of the accessible approach, we get rid of the additional assumption that A is a potential operator by a fixed-point technique. More precisely, for a fixed parameter $\theta > 0$, given any element $w \in K$, consider the auxiliary problem of finding $u \in K$ such that

$$\begin{aligned} (u, v - u)_V + \theta [\Phi(v) - \Phi(u) + \Psi^0(u; v - u)] \\ \geq (w, v - u)_V - \theta [\langle Aw, v - u \rangle - \langle f, v - u \rangle] \quad \forall v \in K. \end{aligned} \quad (4.12)$$

An application of Theorem 4.1 shows that for $\theta > 0$ sufficiently small, the inequality (4.12) admits a unique solution $u \in K$. Moreover, it can be shown that the mapping $w \mapsto u$ is a contraction. By the Banach fixed-point theorem, for $\theta > 0$ sufficiently small, the mapping $w \mapsto u$ has a unique fixed-point, and this unique fixed-point is the unique solution of Problem 4.2. We state the result next; its detailed proof can be found in [50, Section 5.2].

Theorem 4.2. Assume $H(V)$, $H(K)$, $H(A)$, $H(\Phi)$, $H(\Psi)$, $H(f)$, and $\alpha_\Psi < m_A$. Then, Problem 4.2 has a unique solution $u \in K$.

In the last step of the accessible approach, let

$$\Psi(v) = I_\Delta(\psi(\gamma_\psi v)), \quad v \in V \quad (4.13)$$

in (4.11). Denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma_\psi v|_{\mathbb{R}^m}^2) \leq c_\Delta^2 \|v\|_V^2 \quad \forall v \in V. \quad (4.14)$$

It can be shown that $H(\psi)$ implies $H(\Psi)$ with $\alpha_\Psi = \alpha_\psi c_\Delta^2$, cf. Theorem 5.19 in [50].

In most mathematics references (e.g., [81, 96]), the form of a stationary VHI studied is Problem 4.2. We comment that the inequality (4.11) looks simpler than (4.9), however, the form (4.9) arises directly in application problems. Any solution of Problem 4.2 is a solution of Problem 4.1 since ([81, Theorem 3.47])

$$\Psi^0(u; v) \leq I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)).$$

Conversely, if ψ or $-\psi$ is regular in the sense of (2.5), then

$$\Psi^0(u; v) = I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v));$$

as a result, (4.11) and (4.9) coincide, and Problem 4.2 and Problem 4.1 are identical. This is the approach taken in early references on mathematical theory of variational-hemivariational inequalities, e.g., [81].

In our approach, we do not assume the regularity of the function ψ or $-\psi$. Again, we note that a solution of Problem 4.2 is also a solution of Problem 4.1. In addition, it can be shown that a solution of Problem 4.1 is unique. Thus, Problem 4.1 admits a unique solution. The result is stated next, cf. [50, Section 5.4] for details.

Theorem 4.3. *Assume $H(V)$, $H(K)$, $H(A)$, $H(\Phi)$, $H(\psi)$, $H(f)$, and $\alpha_\psi c_\Delta^2 < m_A$. Then Problem 4.1 has a unique solution $u \in K$. Moreover, the solution $u \in K$ depends Lipschitz continuously on $f \in V^*$.*

4.2 Galerkin method for the abstract stationary variational-hemivariational inequality

Since there is no analytic solution formula for a VHI arising in applications, numerical methods are needed to solve the problem. In this subsection, we discuss the numerical solution of Problem 4.1. The numerical method is of Galerkin type. We present a convergence result for the numerical solutions and a Céa-type inequality for error estimation of the numerical solutions. For Problem 4.1, we make the assumptions stated in Theorem 4.3, so as to guarantee that the problem has a unique solution. Let V^h be a finite dimensional subspace of V , $h > 0$ being a spatial discretization parameter. Let K^h be a non-empty, closed and convex subset of V^h . Then, a Galerkin approximation of Problem 4.1 is the following.

Problem 4.4. *Find an element $u^h \in K^h$ such that*

$$\begin{aligned} & \langle Au^h, v^h - u^h \rangle + \Phi(v^h) - \Phi(u^h) + I_\Delta(\psi^0(\gamma_\psi u^h; \gamma_\psi v^h - \gamma_\psi u^h)) \\ & \geq \langle f, v^h - u^h \rangle \quad \forall v^h \in K^h. \end{aligned} \quad (4.15)$$

For the well-posedness of Problem 4.4, we can apply Theorem 4.3 which is valid in the setting of finite-dimensional spaces as well. For completeness, we state the result formally as a theorem.

Theorem 4.4. *Keep the assumptions stated in Theorem 4.3. Let V^h be a finite-dimensional subspace of V and let K^h be a non-empty, closed and convex subset of V^h . Then Problem 4.4 has a unique solution.*

The approximation is called external if $K^h \not\subset K$, and is internal if $K^h \subset K$. In [64], the internal approximation with the choice $K^h = V^h \cap K$ is considered for Problem 4.1.

The following convergence result is proved in [52, Section 4.3] and it follows [65]. An important point about this convergence result is that we do not assume any solution regularity other than the basic regularity $u \in V$ guaranteed in the well-posedness result, namely, Theorem 4.3.

Theorem 4.5. *Keep the assumptions stated in Theorem 4.3. Moreover, assume V^h is a finite-dimensional subspace of V , K^h is a non-empty, closed and convex subset of V^h , and*

$$v^h \in K^h \text{ and } v^h \rightharpoonup v \text{ in } V \text{ imply } v \in K; \quad (4.16)$$

$$\forall v \in K, \exists v^h \in K^h \text{ such that } v^h \rightarrow v \text{ in } V \text{ as } h \rightarrow 0. \quad (4.17)$$

Let u and u^h be the solutions of Problem 4.1 and Problem 4.4, respectively. Then,

$$u^h \rightarrow u \text{ in } V \text{ as } h \rightarrow 0. \quad (4.18)$$

Theorem 4.3 and Theorem 4.5 are rather general, and here we consider two special cases.

First, we consider the case of a HVI with the choice $\Phi \equiv 0$ in Problem 4.1.

Problem 4.5. *Find an element $u \in K$ such that*

$$\langle Au, v - u \rangle + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi u)) \geq \langle f, v - u \rangle \quad \forall v \in K. \quad (4.19)$$

The corresponding numerical method Problem 4.4 takes the following form.

Problem 4.6. *Find an element $u^h \in K^h$ such that*

$$\langle Au^h, v^h - u^h \rangle + I_\Delta(\psi^0(\gamma_\psi u^h; \gamma_\psi v^h - \gamma_\psi u^h)) \geq \langle f, v^h - u^h \rangle \quad \forall v^h \in K^h. \quad (4.20)$$

Theorem 4.6. *Assume $H(V)$, $H(K)$, $H(A)$, $H(\psi)$, $H(f)$, and $\alpha_\psi c_\Delta^2 < m_A$. Moreover, assume V^h is a finite-dimensional subspace of V , K^h is a non-empty, closed and convex subset of V^h , and (4.16)–(4.17) hold. Then, Problem 4.5 admits a unique solution $u \in K$, Problem 4.6 admits a unique solution $u^h \in K^h$, and we have the convergence:*

$$u^h \rightarrow u \text{ in } V \text{ as } h \rightarrow 0.$$

As another particular case, we consider a VI, obtained from Problem 4.1 by setting $\psi \equiv 0$.

Problem 4.7. *Find an element $u \in K$ such that*

$$\langle Au, v - u \rangle + \Phi(v) - \Phi(u) \geq \langle f, v - u \rangle \quad \forall v \in K. \quad (4.21)$$

The numerical method is the following.

Problem 4.8. *Find an element $u^h \in K^h$ such that*

$$\langle Au^h, v^h - u^h \rangle + \Phi(v^h) - \Phi(u^h) \geq \langle f, v^h - u^h \rangle \quad \forall v^h \in K^h. \quad (4.22)$$

Theorem 4.7. *Assume $H(V)$, $H(K)$, $H(A)$, $H(\Phi)$, and $H(f)$. Moreover, assume V^h is a finite-dimensional subspace of V , K^h is a non-empty, closed and convex subset of V^h , and (4.16)–(4.17) hold. Then, Problem 4.7 admits a unique solution $u \in K$, Problem 4.8 admits a unique solution $u^h \in K^h$, and we have the convergence:*

$$u^h \rightarrow u \text{ in } V \text{ as } h \rightarrow 0.$$

For error estimates of the numerical solution defined by Problem 4.4 for the approximation of the solution of Problem 4.1, let $v \in K$ and $v^h \in K^h$ be arbitrary and define

$$R_u(v, w) := \langle Au, v - w \rangle + \Phi(v) - \Phi(w) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v - \gamma_\psi w)) - \langle f, v - w \rangle, \quad (4.23)$$

The following result is proved as Theorem 7 in [52].

Theorem 4.8. *Assume $H(K)$, $H(A)$, $H(\Phi)$, $H(\psi)$, $H(f)$, and $\alpha_\psi c_\Delta^2 < m_A$. Then for the solution u of Problem 4.1 and the solution u^h of Problem 4.4, we have the Céa-type inequality*

$$\|u - u^h\|_V^2 \leq c \inf_{v^h \in K^h} \left[\|u - v^h\|_V^2 + \|\gamma_\psi(u - v^h)\|_{V_\psi} + R_u(v^h, u) \right] + c \inf_{v \in K} R_u(v, u^h). \quad (4.24)$$

In the case of an internal approximation, $K^h \subset K$, and

$$\inf_{v \in K} R_u(v, u^h) = 0.$$

Then, the Céa-type inequality (4.24) simplifies to

$$\|u - u^h\|_V^2 \leq c \inf_{v^h \in K^h} \left[\|u - v^h\|_V^2 + \|\gamma_\psi(u - v^h)\|_{V_\psi} + R_u(v^h, u) \right]. \quad (4.25)$$

We note that (4.24) and (4.25) are generalizations of the Céa-type inequality from the numerical solution of VIs (cf. [32, 61, 74]) to that of VHIs. To proceed further, we need to bound the residual term (4.23) and this depends on the problem to be solved.

5 Studies of the Sample Variational-Hemivariational Inequality

In this section, we return to Problem 3.3 by applying the theoretical results reviewed in Section 4. We first explore the solution existence and uniqueness, then introduce a linear finite element method to solve the problem and present an optimal order error estimate under certain solution regularity assumptions. Details can be found in [52].

Let $\lambda_\nu > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in \mathbf{V}, \quad \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx = \lambda \int_{\Gamma_C} u_\nu v_\nu \, ds \quad \forall \mathbf{v} \in \mathbf{V}.$$

Then, we have the trace inequality

$$\|v_\nu\|_{L^2(\Gamma_C)} \leq \lambda_\nu^{-1/2} \|\mathbf{v}\|_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}.$$

By applying Theorem 4.3, an existence and uniqueness result can be established for Problem 3.3.

Theorem 5.1. *Assume (3.11), (3.12), (3.19), $f_b \geq 0$, and $\alpha_{\psi_\nu} \lambda_\nu^{-1} < m_\varepsilon$. Then Problem 3.3 has a unique solution $\mathbf{u} \in \mathbf{V}$.*

Theorem 5.1 provides the existence of a unique displacement field $\mathbf{u} \in \mathbf{V}$ of the contact problem. The stress field $\boldsymbol{\sigma} \in \mathbb{Q}$ is uniquely determined by using the constitutive law (3.7).

Then, we turn to the discretization of Problem 3.3 using the finite element method. For simplicity, assume Ω is a polygonal domain ($d = 2$) or a polyhedral domain ($d = 3$). We express the three parts Γ_D , Γ_N , Γ_C of the boundary as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_Z} = \cup_{i=1}^{i_Z} \Gamma_{Z,i}, \quad Z = D, N, C.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$ into triangles ($d = 2$) or tetrahedrons ($d = 3$) that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{Z,i}$, $1 \leq i \leq i_Z$, $Z = D, N, C$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{Z,i}$ has a positive measure with respect to $\Gamma_{Z,i}$, then the side ($d = 2$) or the face ($d = 3$) lies entirely in $\Gamma_{Z,i}$. The corresponding linear finite element space is

$$\mathbf{V}^h = \left\{ \mathbf{v}^h \in C(\bar{\Omega})^d \mid \mathbf{v}^h|_T \in \mathbb{P}_1(T)^d \text{ for } T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \overline{\Gamma_D} \right\}. \quad (5.1)$$

The finite element approximation of Problem 3.3 is the following.

Problem 5.1. Find a displacement field $\mathbf{u}^h \in \mathbf{V}^h$ such that

$$\begin{aligned} & \left(\mathcal{E}(\varepsilon(\mathbf{u}^h)), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}^h) \right)_{\mathbb{Q}} + I_{\Gamma_C}(f_b |\mathbf{v}_\tau^h|) - I_{\Gamma_C}(f_b |\mathbf{u}_\tau^h|) \\ & + I_{\Gamma_C}(\psi_\nu^0(u_\nu^h; v_\nu^h - u_\nu^h)) \geq \langle \mathbf{f}, \mathbf{v}^h - \mathbf{u}^h \rangle \quad \forall \mathbf{v}^h \in \mathbf{V}^h. \end{aligned} \quad (5.2)$$

Similar to Theorem 5.1, we know that Problem 5.1 admits a unique solution $\mathbf{u}^h \in \mathbf{V}^h$. Convergence of the finite element solution follows from Theorem 4.5:

$$\|\mathbf{u}^h - \mathbf{u}\|_{\mathbf{V}} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

We comment that the two conditions (4.16)–(4.17) are valid with $K = \mathbf{V}$, $K^h = \mathbf{V}^h$ for the finite element space (5.1).

For an error analysis, we start with

$$\|\mathbf{u} - \mathbf{u}^h\|_{\mathbf{V}}^2 \leq c \inf_{\mathbf{v}^h \in \mathbf{V}^h} \left[\|\mathbf{u} - \mathbf{v}^h\|_{\mathbf{V}}^2 + \|u_\nu - v_\nu^h\|_{L^2(\Gamma_C)} + R_{\mathbf{u}}(\mathbf{v}^h, \mathbf{u}) \right], \quad (5.3)$$

where the residual-type term as defined in (4.23) is

$$\begin{aligned} R_{\mathbf{u}}(\mathbf{v}^h, \mathbf{u}) &= (\mathcal{E}(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v}^h - \mathbf{u}))_{\mathbb{Q}} + I_{\Gamma_C}(f_b(|\mathbf{v}_\tau^h| - |\mathbf{u}_\tau|)) \\ &+ I_{\Gamma_C}(\psi_\nu^0(u_\nu; v_\nu^h - u_\nu)) - \langle \mathbf{f}, \mathbf{v}^h - \mathbf{u} \rangle. \end{aligned} \quad (5.4)$$

To derive an error bound, we need to make solution regularity assumptions:

$$\mathbf{u} \in H^2(\Omega; \mathbb{R}^d), \quad \boldsymbol{\sigma} = \mathcal{E}(\varepsilon(\mathbf{u})) \in H^1(\Omega; \mathbb{S}^d), \quad \mathbf{u}|_{\Gamma_{C,i}} \in H^2(\Gamma_{C,i}; \mathbb{R}^d), \quad 1 \leq i \leq i_C. \quad (5.5)$$

For homogeneous materials, $\mathcal{E}$ does not depend on the spatial variable $\mathbf{x}$ and the second regularity is a consequence of the first one in (5.5). Under the first two regularity assumptions in (5.5), it can be shown that the pointwise relations hold:

$$\operatorname{div} \mathcal{E}(\varepsilon(\mathbf{u})) + \mathbf{f}_0 = \mathbf{0} \quad \text{a.e. in } \Omega, \quad (5.6)$$

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{f}_2 \quad \text{a.e. on } \Gamma_N. \quad (5.7)$$

Then,

$$R_{\mathbf{u}}(\mathbf{v}^h, \mathbf{u}) = \int_{\Gamma_C} \left[\boldsymbol{\sigma} \boldsymbol{\nu} \cdot (\mathbf{v}^h - \mathbf{u}) + f_b(|\mathbf{v}_\tau^h| - |\mathbf{u}_\tau|) + \psi_\nu^0(u_\nu; v_\nu^h - u_\nu) \right] ds,$$

and we can bound

$$|R_{\mathbf{u}}(\mathbf{v}^h, \mathbf{u})| \leq c \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_C; \mathbb{R}^d)}. \quad (5.8)$$

So, from (5.3), we can derive the Céa-type inequality

$$\|\mathbf{u} - \mathbf{u}^h\|_{\mathbf{V}} \leq c \inf_{\mathbf{v}^h \in \mathbf{V}^h} \left[\|\mathbf{u} - \mathbf{v}^h\|_{\mathbf{V}} + \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_C; \mathbb{R}^d)}^{1/2} \right]. \quad (5.9)$$

Due to the first regularity condition in (5.5), by the Sobolev embedding $H^2(\Omega) \subset C(\bar{\Omega})$ valid for $d \leq 3$, we know that $\mathbf{u} \in C(\bar{\Omega}; \mathbb{R}^d)$ and so its finite element interpolant $\Pi^h \mathbf{u} \in \mathbf{V}^h$ is well defined. Moreover, the following error estimate holds (cf. [5, 12, 21]): for some constant $c > 0$ independent of h ,

$$\|\mathbf{u} - \Pi^h \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^d)} + h \|\mathbf{u} - \Pi^h \mathbf{u}\|_{H^1(\Omega; \mathbb{R}^d)} \leq c \|\mathbf{u}\|_{H^2(\Omega; \mathbb{R}^d)}, \quad (5.10)$$

and

$$\|\mathbf{u} - \Pi^h \mathbf{u}\|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq c h^2. \quad (5.11)$$

Then we derive from (5.9) the following optimal order error bound

$$\|\mathbf{u} - \mathbf{u}^h\|_{\mathbf{V}} \leq c \left[\|\mathbf{u} - \Pi^h \mathbf{u}\|_{\mathbf{V}} + \|\mathbf{u} - \Pi^h \mathbf{u}\|_{L^2(\Gamma_C; \mathbb{R}^d)}^{1/2} \right] \leq c h, \quad (5.12)$$

where the constant c depends on the quantities $\|\mathbf{u}\|_{H^2(\Omega; \mathbb{R}^d)}$, $\|\boldsymbol{\sigma} \boldsymbol{\nu}\|_{L^2(\Gamma_C; \mathbb{R}^d)}$ and $\|\mathbf{u}\|_{H^2(\Gamma_{C,i}; \mathbb{R}^d)}$ for $1 \leq i \leq i_C$.

The reader is referred to [52] for numerical examples providing numerical convergence orders that match the error estimate (5.12).

We comment that similar results hold for Problem 3.2 and Problem 3.1 by applying Theorem 4.3 and Theorem 4.8 for the special cases of a VI and a VE.

6 Mixed Variational-Hemivariational Inequalities in Fluid Mechanics

In the previous sections, we considered VHIs from contact mechanics. In this section, we consider sample VHIs in fluid mechanics. Since Fujita's pioneering work [39, 40], there has been steady progress on the modeling, mathematical analysis and numerical approximation of boundary or initial-boundary value problems for flows of viscous incompressible fluid involving nonsmooth slip or leak boundary conditions. When the nonsmooth boundary condition is of monotone type, the mathematical formulation of the problem is a mixed variational inequality. A large number of papers have been published on well-posedness analysis and the numerical solution of Stokes and Navier-Stokes variational inequalities. The reader may consult [50, Chapter 8] for some sample references on this topic. When the nonsmooth boundary condition is allowed to be of nonmonotone type, the weak formulation of the problem is a mixed hemivariational inequality. Various VHIs arising in fluid mechanics have been studied in the literature, e.g., [18, 58, 79, 80, 99]. In this section, we present two mathematical models: a Stokes HVI and a Navier-Stokes HVI.

Let $\Omega \subset \mathbb{R}^d$ ($d \leq 3$ in applications) be a Lipschitz domain. Its boundary $\Gamma = \partial\Omega$ is decomposed into two parts: $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_S$ such that $|\Gamma_D| > 0$, $|\Gamma_S| > 0$, and $\Gamma_D \cap \Gamma_S = \emptyset$. We will impose a Dirichlet boundary condition on Γ_D and a slip boundary condition of friction type on Γ_S . Denote by $\boldsymbol{\nu}$ the unit outward normal to Γ . For a vector-valued function $\mathbf{u}$ on the boundary, let $u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}$ and $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$ be the normal component and the tangential component, respectively. Let $\mu > 0$ be the viscosity coefficient, and let $\mathbf{f}$ be the source function.

The pointwise relations of a sample Stokes HVI are

$$-\operatorname{div}(2\mu \boldsymbol{\varepsilon}(\mathbf{u})) + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (6.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (6.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (6.3)$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_S. \quad (6.4)$$

The condition $u_\nu = 0$ in (6.4) means that the fluid can not pass through Γ_S outside the domain, and this condition is usually called the no-leak condition. The second part in (6.4) represents a friction condition, relating the frictional force $\boldsymbol{\sigma}_\tau$ with the tangential velocity $\mathbf{u}_\tau$. In (6.4), the function $\psi_\tau: \mathbb{R}^d \rightarrow \mathbb{R}$ is locally Lipschitz continuous. We assume the following properties on the function ψ_τ :

$$\left\{ \begin{array}{l} \text{(a) } \psi_\tau(\cdot) \text{ is locally Lipschitz on } \mathbb{R}^d; \\ \text{(b) there exist constants } \bar{c}_0, \bar{c}_1 \geq 0 \text{ such that} \\ \quad |\partial\psi_\tau(\mathbf{z})| \leq \bar{c}_0 + \bar{c}_1 |\mathbf{z}| \quad \forall \mathbf{z} \in \mathbb{R}^d; \\ \text{(c) there exists a constant } \alpha_{\psi_\tau} \geq 0 \text{ such that} \\ \quad \psi_\tau^0(\mathbf{z}_1; \mathbf{z}_2 - \mathbf{z}_1) + \psi_\tau^0(\mathbf{z}_2; \mathbf{z}_1 - \mathbf{z}_2) \leq \alpha_{\psi_\tau} |\mathbf{z}_1 - \mathbf{z}_2|^2 \quad \forall \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^d. \end{array} \right. \quad (6.5)$$

We comment that it is possible to consider the more general case $\psi_\tau: \Gamma_S \times \mathbb{R}^d \rightarrow \mathbb{R}$ where ψ_τ depends on the spatial variable $\mathbf{x}$.

In this section, we define the function space

$$\mathbf{V} = \{ \mathbf{v} \in \mathbf{H}^1(\Omega) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, \ v_\nu = 0 \text{ on } \Gamma_S \}, \quad (6.6)$$

for the velocity field, where $\mathbf{H}^1(\Omega) = H^1(\Omega; \mathbb{R}^d)$. As a consequence of Korn's inequality, the quantity

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega; \mathbb{S}^d)} := \left(\int_\Omega |\boldsymbol{\varepsilon}(\mathbf{v})|^2 dx \right)^{\frac{1}{2}}$$

defines a norm for $\mathbf{v} \in \mathbf{V}$ and it is equivalent to the standard $\mathbf{H}^1(\Omega)$ -norm on $\mathbf{V}$. We use

$$\|\cdot\|_{\mathbf{V}} = \|\boldsymbol{\varepsilon}(\cdot)\|_{L^2(\Omega; \mathbb{S}^d)} \quad (6.7)$$

for the norm on $\mathbf{V}$. For the pressure variable, we use the space

$$Q = L_0^2(\Omega) = \{ q \in L^2(\Omega) \mid I_\Omega(q) = 0 \}. \quad (6.8)$$

This is a Hilbert space with the standard inner product

$$(p, q)_{0, \Omega} := \int_\Omega p q dx$$

and the corresponding $\|\cdot\|_{0, \Omega}$ -norm.

Define the following forms:

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) dx \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \quad (6.9)$$

$$b(\mathbf{v}, q) = \int_\Omega q \operatorname{div} \mathbf{v} dx \quad \forall \mathbf{v} \in \mathbf{V}, \ q \in Q. \quad (6.10)$$

The weak formulation of the problem (6.1)–(6.4) is derived with the standard approach.

Problem 6.1. Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + I_{\Gamma_S}(\psi_\tau^0(\mathbf{u}_\tau; \mathbf{v}_\tau)) \geq \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (6.11)$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in Q. \quad (6.12)$$

Let $\lambda_\tau > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in \mathbf{V}, \quad \int_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx = \lambda \int_{\Gamma_S} \mathbf{u}_\tau \cdot \mathbf{v}_\tau \, ds \quad \forall \mathbf{v} \in \mathbf{V}. \quad (6.13)$$

Then, we have the trace inequality

$$\|\mathbf{v}_\tau\|_{L^2(\Gamma_S; \mathbb{R}^d)} \leq \lambda_\tau^{-1/2} \|\mathbf{v}\|_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}.$$

The following well-posedness result is labelled as Theorem 8.22 in [50].

Theorem 6.1. Assume (6.5) and $\alpha_{\psi_\tau} < 2\mu\lambda_\tau$. Then, for any $\mathbf{f} \in \mathbf{V}^*$, Problem 6.1 has a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$ which depends Lipschitz continuously on $\mathbf{f} \in \mathbf{V}^*$.

We skip details of discussions of the numerical solution of Problem 6.1 in this paper, and refer the reader to [33] on a mixed finite element method and to [75] on stabilized mixed finite element methods to solve Problem 6.1.

We now turn to a sample Navier-Stokes HVI. The pointwise formulation of the problem is

$$-\operatorname{div}(2\mu \varepsilon(\mathbf{u})) + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (6.14)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (6.15)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (6.16)$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_S. \quad (6.17)$$

Use the function space $\mathbf{V}$ defined in (6.6) for the velocity, and the function space Q defined in (6.8) for the pressure. Use the forms $a(\mathbf{u}, \mathbf{v})$ defined in (6.9), $b(\mathbf{v}, q)$ defined in (6.10). Furthermore, define

$$d(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, dx \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}. \quad (6.18)$$

We continue to assume $\mathbf{f} \in \mathbf{V}^*$ and (6.5). Then, the weak formulation of the problem is

Problem 6.2. Find $\mathbf{u} \in \mathbf{V}$ and $p \in Q$ such that

$$a(\mathbf{u}, \mathbf{v}) + d(\mathbf{u}, \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + I_{\Gamma_S}(\psi_\tau^0(\mathbf{u}_\tau; \mathbf{v}_\tau)) \geq \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (6.19)$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in Q. \quad (6.20)$$

The following well-posedness result on Problem 6.2 is shown in [50, Section 8.5].

Theorem 6.2. Assume (6.5) and

$$\alpha_{\psi_\tau} \lambda_\tau^{-1} < 2\mu, \quad \alpha_{\psi_\tau} \lambda_\tau^{-1} + c_d M_{\mathbf{f}} < 2\mu,$$

where $c_d > 0$ is a constant in the boundedness inequality

$$|d(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c_d \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}} \|\mathbf{w}\|_{\mathbf{V}} \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V},$$

the constant $M_{\mathbf{f}}$ is defined by

$$M_{\mathbf{f}} = \frac{c_0 \lambda_\tau^{-1/2} |\Gamma_S|^{1/2} + \|\mathbf{f}\|_{\mathbf{V}^*}}{2\mu - \alpha_{\psi_\tau} \lambda_\tau^{-1}}$$

and $\lambda_\tau > 0$ is the smallest eigenvalue of the eigenvalue problem (6.13). Then, Problem 6.2 admits a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$, $\|\mathbf{u}\|_{\mathbf{V}} \leq M_{\mathbf{f}}$, and $(\mathbf{u}, p) \in \mathbf{V} \times Q$ depends Lipschitz continuously on $\mathbf{f} \in \mathbf{V}^*$.

Once again, we skip a detailed discussion of the numerical solution of Problem 6.2 in this paper, and refer the reader to [51] on a mixed finite element method and to [53] on stabilized mixed finite element methods to solve Problem 6.2.

Finally, we mention that research on VHIs for problems in fluid mechanics is not limited to the standard Stokes and Navier-Stokes equations. E.g., HVIs are analyzed together with their finite element solutions for incompressible fluid flows with damping in [59] for the Stokes equations, and in [108] for the Navier-Stokes equations. In [3], a HVI is studied for incompressible fluid flows with damping and pumping effects.

7 Miscellaneous Remarks

In the previous sections, we discussed the finite element method to solve stationary and mixed VHIs. Other numerical methods have been applied to solve VHIs as well. E.g., the discontinuous Galerkin method is applied to solve a HVI for semipermeable media in [103] and to solve a contact problem in [104]. Take the virtual element method (VEM) as another example. The VEM was first proposed and analyzed in [10, 11]. The method has since been applied to a wide variety of mathematical models from applications in science and engineering thanks to its strengths in handling complex geometries and problems requiring high-regularity solutions. The VEM was first applied to solve contact problems in [110]. Further applications of the VEM to solve contact problems can be found in a number of publications, e.g., [2, 22, 106, 112]. The VEM was first applied to solve a HVI in [34]. Further references on application of the VEM to solve VHIs are [35–37, 76, 105, 111, 113, 114].

In [8], a numerical method is applied to solve a VHI modeling contact problems for locking materials. An error estimate is derived in the paper; however, derivation of an optimal order error bound remains open.

To solve discretized VHIs, optimization based numerical algorithms are developed in [72] and [85] for stationary VHIs, and in [73] for time-dependent VHIs, all with applications to contact mechanics. A theoretical foundation for the optimization approach is explored in [48], where it is shown that certain special VHIs are equivalent to optimization problems. The possibility of reformulating some VHIs as optimization problems is a starting point to develop deep neural networks to solve those VHIs ([70]). Deep neural network techniques have also been explored to solve an obstacle problem in [19]. For another kind of neural network methods to solve obstacle problems, see [102]. More research is expected on neural network methods to solve VHIs.

Most of the papers on VHIs deal with PDEs of second-order. In [37], a nonconforming virtual element method is studied for a fourth-order HVI in the Kirchhoff plate problem. In [89], an interior penalty virtual element method is developed to solve a fourth-order HVI. In [88], a nonconforming finite element method is developed to solve a fourth-order history-dependent HVI.

Although most of the papers on the numerical solution of VHIs deal with the stationary (i.e., time-independent) case, various papers are available on numerical methods for solving time-dependent VHIs. Some representative publications are as follows. In [7], a hyperbolic HVI arising in a dynamic contact problem is considered. Both spatially semi-discrete schemes and fully discrete schemes are introduced and analyzed. Optimal order error estimates are derived with the use of linear finite elements. In [6], numerical analysis is provided on a parabolic VHI with an application to a contact problem for viscoelastic bodies. In [46], numerical analysis is provided on another parabolic VHI with an application to a dynamic contact problem for viscoelastic bodies. In [9], a fully discrete scheme is applied to solve a coupled system of HVIs for a nonmonotone dynamic contact problem of a non-clamped piezoelectric viscoelastic body.

The term history-dependent (quasi-)variational inequalities first appears in the paper [93]. History-dependent VHIs are first analyzed in [95]. The numerical solution of history-dependent VHIs by the

finite element method for spatial discretizations is studied in [107, 116, 117]. The virtual element method is applied to solve a history-dependent VHI in [115], and is applied to solve a history-dependent mixed VHI in [77], both concerning applications in contact problems.

In [118], numerical analysis for a system of fractional differential hemivariational inequalities in the consideration of a thermoviscoelastic frictional contact problem involving time-fractional order operators and long memory effects. In [98], numerical analysis for a coupled system of a dynamic HVI, a parabolic VI and a parabolic PDE that describes a thermoviscoelastic contact problem with damage and long memory.

Besides contact mechanics and fluid mechanics, HVIs appear also in other fields of applications. As an example, a HVI from Bean's model for an irreversible and hysteretic magnetization process of high-temperature superconductors in a magnetic field is studied in [54].

This paper provides an introduction of basic mathematical theory and numerical analysis of VHIs, focusing on the stationary ones. Well-posedness analysis of VHIs is conducted with an approach that does not require knowledge of abstract theory of pseudomonotone operators. The main idea of this accessible approach for the study of stationary VHIs is explained in the paper. We hope this helps to attract more researchers in applied and computational mathematics, engineering to the exciting and challenging area of VHIs.

A basic theory is sketched for the numerical solution of stationary VHIs. The numerical scheme is constructed within the framework of Galerkin methods, and in particular, by the finite element method. Convergence of the numerical solutions is explored, and derivation of error estimates for the numerical solutions is demonstrated through the approximation of a stationary VHI in contact mechanics. Besides the finite element method, references are also provided for the use of the virtual element method and the discontinuous Galerkin method. Further efforts are needed for the development of efficient and effective solution algorithms to solve the discretized VHIs and for the analysis of such algorithms. It is even possible to develop machine learning methods to solve VHIs.

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