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Survey Article

Shepard Operators Revisited: Linear and Nonlinear Approaches, Accelerated Convergence, and Applications

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Abstract: Shepard operators are important in scattered data approximation. They were first introduced as linear positive operators for multivariate interpolation, and later developed into a broader class of approximation tools. In this survey, we review the classical Shepard operator, its multivariate forms, and several modifications aimed at improving convergence. We consider both linear and nonlinear extensions of Shepard operators.

We also give some numerical results and graphical illustrations supporting the theoretical results. We aim to offer a unified and current reference for researchers in approximation theory, scattered data methods, and operator-based interpolation.

Keywords: Shepard operators; Taylor polynomials; Lagrange polynomials; Bernoulli polynomials; Lidstone polynomials; interpolation; max-product operations

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1 Introduction

Shepard operators were first introduced in 1968 by Donald Shepard [1] and later became known in the literature as the inverse distance weighting method (see, for instance, [2–4]). Since then, they have attracted constant attention in approximation theory and related areas of mathematics (see [5–28]). Researchers have also studied many variants and generalizations of the classical form, often with the goal of improving accuracy and convergence (see [29–43]).

Shepard-type methods have found broad applications not only in mathematics but also in practice. Especially, they are used in computer graphics, geographic information systems, image processing, and many other fields where scattered data must be interpolated (see, for instance, [44–49]).

We now begin with the definition of operators. Let $[a,b]$ be a given interval. Suppose that we have interpolation (sample) points

$$x_0, x_1, \ldots, x_n \in [a,b],$$

and the corresponding function values

$$f(x_0), f(x_1), \ldots, f(x_n).$$

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The classical Shepard operators [1] are defined by

$$S_{n,\lambda}(f;x) := \sum_{k=0}^{n} W_{n,k}(x,\lambda)\, f(x_k), \qquad x \in [a,b], \tag{1.1}$$

where the weights are given by

$$W_{n,k}(x,\lambda) := \frac{|x-x_k|^{-\lambda}}{\sum_{i=0}^{n} |x-x_i|^{-\lambda}}, \qquad \lambda > 0.$$

Note that the weight functions are not differentiable at the interpolation points when $\lambda \leq 1$. For this reason, the case $\lambda > 1$ is usually considered. Throughout this work, we sometimes consider arbitrarily scattered interpolation points (nodes) in an interval $[a,b]$, and sometimes the following equally spaced interpolation points in $[0,1]$:

$$x_k = k/n, \qquad k = 0,1,\ldots,n,$$

Then, we know the following fundamental properties:

- **Interpolation property:** $S_{n,\lambda}(f;x_k) = f(x_k)$ for each $k = 0,1,...,n$.
- **Positivity:** $S_{n,\lambda}(f) \geq 0$ whenever $f \geq 0$.
- **Linearity:** $S_{n,\lambda}(\alpha f + \beta g) = \alpha S_{n,\lambda}(f) + \beta S_{n,\lambda}(g)$ for every function f, g defined on $[0,1]$ and for all $\alpha, \beta \in \mathbb{R}$.
- **Partition of unity:** $\sum_{k=0}^{n} W_{k,\lambda}(x) = 1$ for all $x \in [0,1]$.
- **Approximation behavior:** $S_{n,\lambda}(f) \rightrightarrows f$ on $[0,1]$ for all $f \in C[0,1]$, where $C[0,1]$ denotes the space of all continuous functions on $[0,1]$ and $\rightrightarrows$ denotes the uniform convergence.
- **Localization:** Larger values of the parameter λ increase the influence of nearby nodes and reduce the effect of distant ones.
- **Degree of exactness:** $dex(S_{n,\lambda}) = 0$, where the symbol dex denotes the degree of exactness in the sense that Shepard operators reproduce only constants, i.e., the equality $S_{n,\lambda}(f) = f$ holds only for constant functions.

A nonlinear version of Shepard operators was first introduced by Bede et. al. [11] as follows:

$$P_{n,\lambda}(f;x) := \frac{\bigvee\limits_{k=0}^{n} f\left(\frac{k}{n}\right) |x - \frac{k}{n}|^{-\lambda}}{\bigvee\limits_{k=0}^{n} |x - \frac{k}{n}|^{-\lambda}}, \quad \lambda > 0, \quad x \in [0,1], \tag{1.2}$$

where the symbol $\bigvee$ denotes the maximum operation. These operators are known in the literature as max-product Shepard operators. Observe that the max-product Shepard operators also show a performance very close to the properties mentioned above. However, in this case, the positivity of continuous functions is required for uniform convergence, and also instead of linearity, these operators satisfy the following pseudo-linearity property.

- **Pseudo-linearity:** $P_{n,\lambda}(\alpha f \vee \beta g) = \alpha S_{n,\lambda}(f) \vee \beta S_{n,\lambda}(g)$ for every nonnegative function f, g defined on $[0,1]$ and for all $\alpha, \beta \geq 0$.

The max-product Shepard operators provide easier computation than their linear counterpart, and they exhibit better approximation performance for suitable classes of functions.

The rates of convergence of the Shepard operators in linear and nonlinear cases are given below, respectively:

Theorem 1.1. (see [1, 5, 42, 43]) *Let f be Lipschitz continuous on $[0,1]$. Then, for every $x \in [0,1]$ and $n \in \mathbb{N}$,*

$$|S_{n,\lambda}(f;x) - f(x)| = \begin{cases} \mathcal{O}\left(1/\log n\right), & \text{if } \lambda = 1 \\ \mathcal{O}\left(1/n^{\lambda-1}\right), & \text{if } 1 < \lambda < 2 \\ \mathcal{O}\left((\log n)/n\right) & \text{if } \lambda = 2 \\ \mathcal{O}\left(1/n\right), & \text{if } \lambda > 2. \end{cases}$$

holds

Theorem 1.2. (see [42, 43]) *Let f be Lipschitz continuous and nonnegative on $[0,1]$. Then, for every $x \in [0,1]$ and $n \in \mathbb{N}$,*

$$|P_{n,\lambda}(f;x) - f(x)| = \begin{cases} \mathcal{O}\left(1/n^{\lambda}\right), & \text{if } 0 < \lambda < 1 \\ \mathcal{O}\left(1/n\right), & \text{if } \lambda \geq 1. \end{cases}$$

holds

In this work, we give an overview of recent hybrid methods from the literature that aim to improve Theorems 1.1 and 1.2. In Sections 2–5, we consider the combined Shepard-Taylor, Shepard-Lagrange, Shepard-Lidstone, and Shepard-Bernoulli operators, respectively, and recall their superior approximation properties. Section 6 contains numerical data and graphics that support the theoretical results. Finally, in Section 7, we provide concluding remarks and possible future perspectives.

2 Combined Shepard-Taylor Operators

In this section, we will consider the multivariate cases of (1.1), (1.2) and use the notation in [29] and [42]. For a given $m, n \in \mathbb{N}$, define the sets $\mathbf{K}$ and Ω_n as follows:

$$\mathbf{K} := [0,1]^m = \left\{\mathbf{x} = (x_1, x_2, ..., x_m) \in \mathbb{R}^m : x_i \in [0,1],\ i = 1,2,...,m\right\},$$

$$\Omega_n := \left\{\mathbf{k} = (k_1, k_2, ..., k_m) \in \mathbb{N}^m : k_i \in \{0,1,...,n\},\ i = 1,2,...,m\right\}.$$

Then, considering the interpolation points of $\mathbf{K}$

$$\mathbf{x}_{\mathbf{k},n} = \left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_m}{n}\right) \quad \text{for} \quad \mathbf{k} \in \Omega_n,$$

we define the corresponding kernel functions (for $\lambda > 0$), respectively:

$$s_{n,\mathbf{k}}(\mathbf{x},\lambda) := \begin{cases} \delta_{\mathbf{k},\mathbf{r}}, & \text{if } \mathbf{x} = \mathbf{x}_{\mathbf{r},n} \ \ (\mathbf{r} \in \Omega_n) \\ \frac{\left|\mathbf{x}-\mathbf{x}_{\mathbf{k},n}\right|_m^{-\lambda}}{\sum_{\mathbf{k}\in\Omega_n}\left|\mathbf{x}-\mathbf{x}_{\mathbf{k},n}\right|_m^{-\lambda}}, & \text{otherwise} \end{cases} \tag{2.1}$$

and

$$p_{n,\mathbf{k}}(\mathbf{x},\lambda) := \begin{cases} \delta_{\mathbf{k},\mathbf{r}}, & \text{if } \mathbf{x} = \mathbf{x}_{\mathbf{r},n} \ \ (\mathbf{r} \in \Omega_n) \\ \frac{\left|\mathbf{x}-\mathbf{x}_{\mathbf{k},n}\right|_m^{-\lambda}}{\bigvee_{\mathbf{k}\in\Omega_n}\left|\mathbf{x}-\mathbf{x}_{\mathbf{k},n}\right|_m^{-\lambda}}, & \text{otherwise,} \end{cases} \tag{2.2}$$

where $\delta_{\mathbf{k},\mathbf{r}}$ is the Kronecker delta given by

$$\delta_{\mathbf{k},\mathbf{r}} = \begin{cases} 1, & \text{if } \mathbf{k} = \mathbf{r}, \\ 0, & \text{if } \mathbf{k} \neq \mathbf{r}. \end{cases}$$

Using the above kernel functions, the multivariate linear and nonlinear (max-product) Shepard operators are defined respectively by

$$S_{n,\lambda}(f;\mathbf{x}) := \sum_{\mathbf{k}\in\Omega_n} s_{n,\mathbf{k}}(\mathbf{x},\lambda) f(\mathbf{x}_{\mathbf{k},n}).$$

and

$$P_{n,\lambda}(f;\mathbf{x}) := \bigvee_{\mathbf{k}\in\Omega_n} p_{n,\mathbf{k}}(\mathbf{x},\lambda) f(\mathbf{x}_{\mathbf{k},n}).$$

Now let $\mathbf{x} = (x_1, x_2, ..., x_m) \in \mathbf{K}$, $\mathbf{u} = (u_1, u_2, ..., u_m) \in \mathbb{N}_0^m$, and $f \in C^r(\mathbf{K})$, the class of the functions $f : \mathbf{K} \to \mathbb{R}$ having continuous r-th order partial derivatives. We also use the following multi-index notation.

$$\begin{aligned} |\mathbf{u}| &= u_1 + u_2 + \cdots + u_m \\ \mathbf{u}! &= u_1! u_2! \cdots u_m! \\ \mathbf{x}^{\mathbf{u}} &= x_1^{u_1} x_2^{u_2} \cdots x_m^{u_m} \\ \partial^{\mathbf{u}} f &= \partial_1^{u_1} \partial_2^{u_2} \cdots \partial_m^{u_m} = \frac{\partial^{|\mathbf{u}|} f}{\partial x_1^{u_1} \partial x_2^{u_2} \cdots \partial x_m^{u_m}}, \quad |\mathbf{u}| \leq r. \end{aligned}$$

Then, the r-th order (combined) linear and max-product Shepard-Taylor operators are defined respectively by

$$ST_{n,\lambda}^{[r]}(f;\mathbf{x}) := \sum_{\mathbf{k}=\mathbf{0}}^{n} s_{n,\mathbf{k}}(\mathbf{x},\lambda) \left(\sum_{|\mathbf{u}|\leq r} \frac{\partial^{\mathbf{u}} f(\mathbf{x}_{\mathbf{k},n})}{\mathbf{u}!} (\mathbf{x} - \mathbf{x}_{\mathbf{k},n})^{\mathbf{u}} \right). \tag{2.3}$$

and

$$PT_{n,\lambda}^{[r]}(f;\mathbf{x}) := \bigvee_{\mathbf{k}=\mathbf{0}}^{n} p_{n,\mathbf{k}}(\mathbf{x},\lambda) \left(\sum_{|\mathbf{u}|\leq r} \frac{\partial^{\mathbf{u}} f(\mathbf{x}_{\mathbf{k},n})}{\mathbf{u}!} (\mathbf{x} - \mathbf{x}_{\mathbf{k},n})^{\mathbf{u}} \right). \tag{2.4}$$

In this case, we know that (2.3) and (2.4) possess the interpolation property and satisfy the following:

$$dex\left(ST_{n,\lambda}^{[r]}\right) = dex\left(PT_{n,\lambda}^{[r]}\right) = r.$$

With this terminology, Farwig proved the next result.

Theorem 2.1. (see [29]) *If the partial derivatives $\partial^{\mathbf{u}} f$ of f of order r ($|\mathbf{u}| = r$) are Lipschitz continuous in $\mathbf{K}$, then, for every $\mathbf{x} \in \mathbf{K}$ and $n \in \mathbb{N}$,*

$$\left| ST_{n,\lambda}^{[r]}(f;\mathbf{x}) - f(\mathbf{x}) \right| = \begin{cases} \mathcal{O}\left(\frac{1}{\log n}\right), & \text{if } \lambda = m \\ \mathcal{O}\left(\frac{1}{n^{\lambda-m}}\right), & \text{if } m < \lambda < m+r+1 \\ \mathcal{O}\left(\frac{\log n}{n^{r+1}}\right), & \text{if } \lambda = m+r+1 \\ \mathcal{O}\left(\frac{1}{n^{r+1}}\right), & \text{if } \lambda > m+r+1 \end{cases}$$

holds.

The max-product version of Theorem 2.1 has recently been proved in [42] as follows.

Theorem 2.2. (see [42]) *If f is nonnegative on $\mathbf{K}$ and the partial derivatives $\partial^{\mathbf{u}} f$ of f of order r ($|\mathbf{u}| = r$) are Lipschitz continuous in $\mathbf{K}$, then, for every $\mathbf{x} \in \mathbf{K}$ and $n \in \mathbb{N}$,*

$$\left| PT^{[r]}_{n,\lambda}(f;\mathbf{x}) - f(\mathbf{x}) \right| = \begin{cases} \mathcal{O}\left(\frac{1}{n^{[\lambda]}}\right), & \text{if } 1 \leq \lambda < r+1 \\ \mathcal{O}\left(\frac{1}{n^{r+1}}\right), & \text{if } \lambda \geq r+1 \end{cases}$$

holds.

A comparison of the above two theorems and the advantages of the nonlinear case are discussed systematically in [42].

3 Combined Shepard-Lagrange Operators

For a function $f : [0,1] \to \mathbb{R}$ and a dataset $\mathcal{A} = \{x_0, x_1, ..., x_r\}$ consisting of $r+1$ distinct nodes from the interval $[0,1]$, we denote the corresponding (unique) Lagrange interpolation polynomial by $L_r[f;\mathcal{A}]$. As usual, this interpolation polynomial is written as follows:

$$L_r[f;\mathcal{A}](x) = \sum_{i=0}^{r} f(x_i) \prod_{0 \leq j \leq r\ (j \neq i)} \frac{x - x_j}{x_i - x_j}.$$

If $f \in C^{r+1}[0,1]$, $r \in \mathbb{N}$, then, we know from the Lagrange remainder theorem (see, for instance, [50]) that, for every $x \in [0,1]$, there exists ξ in the interval spanned by x and x_i, $i = 0,1,...,r$, such that

$$R(x) := f(x) - L_r[f;\mathcal{A}](x) = \frac{f^{(r+1)}(\xi)}{(r+1)!} \prod_{i=0}^{r} (x - x_i),$$

The idea of combining Shepard operators with Lagrange polynomials has been explored in some previous studies (see [33, 37, 43]). In this paper, we will consider the following hybrid models:

$$SL^{[r]}_{n,\lambda}(f;x) := \sum_{k=0}^{n} s_{n,k}(x,\lambda) L_r[f;\mathcal{A}_k](x), \tag{3.1}$$

and

$$PL^{[r]}_{n,\lambda}(f;x) := \sum_{k=0}^{n} p_{n,k}(x,\lambda) L_r[f;\mathcal{A}_k](x), \tag{3.2}$$

where $s_{n,k}(x,\lambda)$ and $p_{n,k}(x,\lambda)$ are univariate case of the kernels in (2.1) and (2.2), respectively. Also, in (3.1) and (3.2), we use the data set $\mathcal{A}_k$, $k = 0,1,...,n$, given by

$$\mathcal{A}_k = \{x_{n,k}, x_{n,k+1}, ..., x_{n,k+r}\},$$

where

$$x_{n,k+j} := \begin{cases} \frac{k+j}{n}, & \text{if } 0 \leq k+j \leq n \\ \frac{k+j-r-1}{n}, & \text{if } n < k+j \leq r. \end{cases}$$

Theorem 3.1. (see [43]) *Let $f \in C^{r+1}[0,1]$ ($r \in \mathbb{N}_0$ with $r < n$). Then, for every $x \in [0,1]$, and $n \in \mathbb{N}$, we have*

$$\left| SL^{[r]}_{n,\lambda}(f;x) - f(x) \right| = \begin{cases} \mathcal{O}(1/\log n), & \text{if } \lambda = 1 \\ \mathcal{O}\left(1/n^{\lambda-1}\right), & \text{if } 1 < \lambda < r+2. \\ \mathcal{O}\left((\log n)/n^{r+1}\right), & \text{if } \lambda = r+2. \\ \mathcal{O}\left(1/n^{r+1}\right), & \text{if } \lambda > r+2. \end{cases}$$

Theorem 3.2. (see [43]) *Let $f \in C^{r+1}[0,1]$ ($r \in \mathbb{N}_0$ with $r < n$) and let f be nonnegative on $[0,1]$. Then, for every $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

$$\left|PL_{n,\lambda}^{[r]}(f;x) - f(x)\right| = \begin{cases} O\left(1/n^{\lambda}\right), & \text{if } 0 < \lambda < r+1. \\ O\left(1/n^{r+1}\right), & \text{if } \lambda \geq r+1. \end{cases}$$

A comparison of Theorems 3.1 and 3.2, together with the benefits of the nonlinear case, was also discussed in [43].

4 Combined Shepard-Bernoulli Operators

The r-th degree Bernoulli polynomials $B_r(x)$ are defined recursively by means of the following relations:

$$\begin{cases} B_0(x) = 1 \\ B_r'(x) = rB_{r-1}(x), \quad r \geq 1 \\ \int_0^1 B_r(x)dx = 0, \quad r \geq 1. \end{cases}$$

Then, for a function $f \in C^r([a,b])$, we may write that

$$f(x) = P_r[f,a,b](x) + R_r[f,a,b](x), \qquad x \in [a,b],$$

In this equality, $P_r[f,a,b](x)$ denotes the polynomial approximant given by

$$P_r[f,a,b](x) = f(a) + \sum_{k=1}^{r} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!} h^{k-1}\Big(f^{(k-1)}(b) - f^{(k-1)}(a)\Big),$$

where $h = b-a$ and $B_k = B_k(0)$ are the Bernoulli numbers. Also, $R_r[f,a,b](x)$ denotes the remainder term evaluated by

$$R_r[f,a,b](x) = \frac{h^{m-1}}{r!} \int_a^b f^{(r)}(t) \left(B_r\left(\frac{b-t}{h}\right) - B_r\left(\frac{x-t-[x-t]}{h}\right)\right) dt.$$

An important property of $P_r[f,a,b]$ is that, as $h \to 0$, it coincides with the Taylor polynomial of degree r at a

From the above considerations, the Shepard-Bernoulli operator and its approximation properties can be stated as follows:

$$SB_{n,\lambda}^{[r]}(f;x) := \sum_{k=0}^{n} s_{n,k}(x,\lambda) P_r[f;x_k,x_{k+1}](x), \tag{4.1}$$

where $x_k = k/n$ ($k = 0,1,...,n$) and $x_{n+1} = (n-1)/n$.

Theorem 4.1. (see [41]) *For every $x \in [0,1]$, $n \in \mathbb{N}$ and $f \in C^r[0,1]$, we have*

$$\left|SB_{n,\lambda}^{[r]}(f;x) - f(x)\right| = \begin{cases} \mathcal{O}\left(1/\log n\right), & \text{if } \lambda = 1 \\ \mathcal{O}\left(1/n^{\lambda-1}\right), & \text{if } 1 < \lambda < r+1. \\ \mathcal{O}\left((\log n)/n^{\lambda-1}\right), & \text{if } \lambda = r+1. \\ \mathcal{O}\left(1/n^{r}\right), & \text{if } \lambda > r+1. \end{cases}$$

For the bivariate extension of the operators in (4.1) and their further approximation behavior, we refer the reader to the works in [35, 36, 41].

5 Combined Shepard-Lidstone Operators

In 1929, Lidstone [51], and later in 1932, Poritsky [52], independently introduced a generalization of the Taylor series. Unlike the classical case, this expansion approximates a function around two points rather than one. Recall that the r-th order Lidstone polynomials $\Lambda_r(x)$ [53] are defined recursively by

$$\begin{cases} \Lambda_0(x) = x \\ \Lambda_r''(x) = \Lambda_{r-1}(x), \quad r \geq 1 \\ \Lambda_r(0) = \Lambda_r(1) = 0, \quad r \geq 1. \end{cases}$$

Now, let $[a, b]$ be any subinterval of $[0, 1]$. Then, for a function $f \in C^{2r}[a, b]$, the relation

$$f(x) = D_r[f, a, b](x) + R_r^D[f, a, b](x)$$

holds, where $D_r[f, a, b](x)$ is the polynomial of degree not greater than $2r - 1$ defined by

$$D_r[f, a, b](x) = \sum_{k=0}^{n-1} h^{2k} \left[\Lambda_k \left(\frac{b-x}{h} \right) f^{(2k)}(a) + \Lambda_k \left(\frac{x-a}{h} \right) f^{(2k)}(b) \right],$$

and $R_r^D[f, a, b](x)$ denotes the remainder term evaluated by

$$\begin{aligned} R_r^D[f, a, b](x) = & \int_a^x f^{(2r)}(t) \frac{(x-t)^{2r-1}}{(2r-1)!} dt \\ & - h^{2r} \sum_{k=0}^{r-1} \Lambda_k \left(\tfrac{x-a}{h} \right) \int_0^1 f^{(2r)}(a+th) \frac{(1-t)^{2r-2k-1}}{(2r-2k-1)!} dt, \end{aligned}$$

where $h = b - a$. From the above considerations, the (combined) Shepard-Lidstone operators are defined by

$$SD_{n,\lambda}^{[r]}(f; x) := \sum_{k=0}^{n} s_{n,k}(x, \lambda) D_r[f; x_k, x_{k+1}](x), \tag{5.1}$$

where $x_k = k/n$, $k = 0, 1, ..., n$ and $x_{n+1} = (n-1)/n$ as stated before. For a detailed account of the main properties of the operators in (5.1) and their bivariate extensions, as well as their superior approximation performance, we refer the reader to [32, 34, 38].

6 Numerical Experiments

We now present graphs that illustrate the approximation results of the Shepard-type operators discussed in the previous sections. In Figures 1–4, we fix the parameters $\lambda = 3$, $r = 1$, and take the values $n = 8$ and $n = 15$. As a test function, we use

$$f(x) = 2 + e^{-2x} \sin(12x), \qquad x \in [0, 1]. \tag{6.1}$$

We then compare the approximation behavior of the considered operators with respect to this function.

In Figure 5, we observe that the approximation improves as r increases. Using the same test function as in (6.1), and fixing $n = 5$, $\lambda = 6$, and also taking $r = 1, 2, 3$, the approximation performance of the Shepard-Taylor type operators is illustrated.

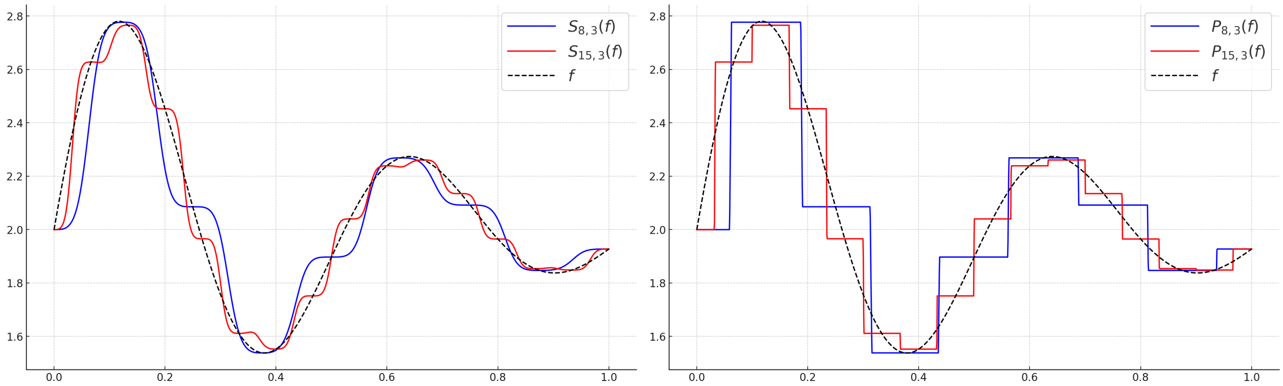

Figure 1: Approximation by the classical Shepard operators (left) and the max-product Shepard operators (right)

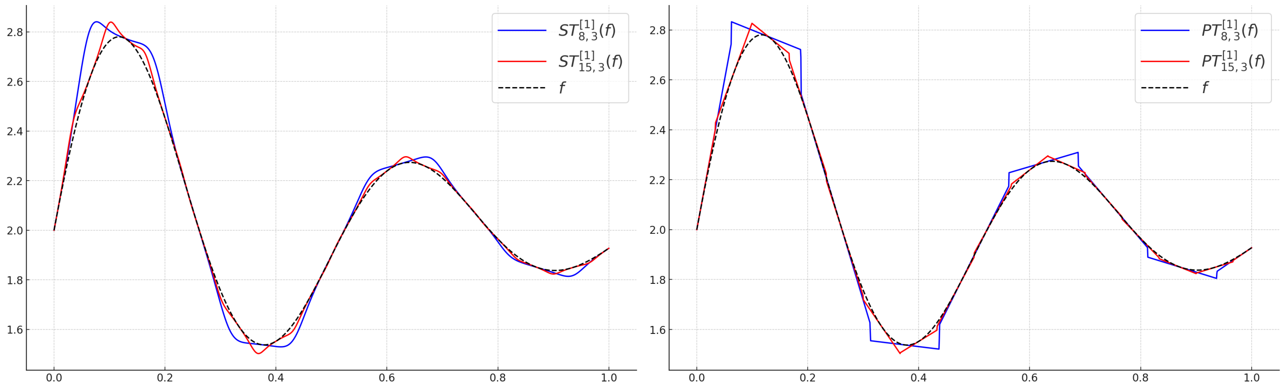

Figure 2: Approximation by the Shepard-Taylor operators (left) and the max-product Shepard-Taylor operators (right)

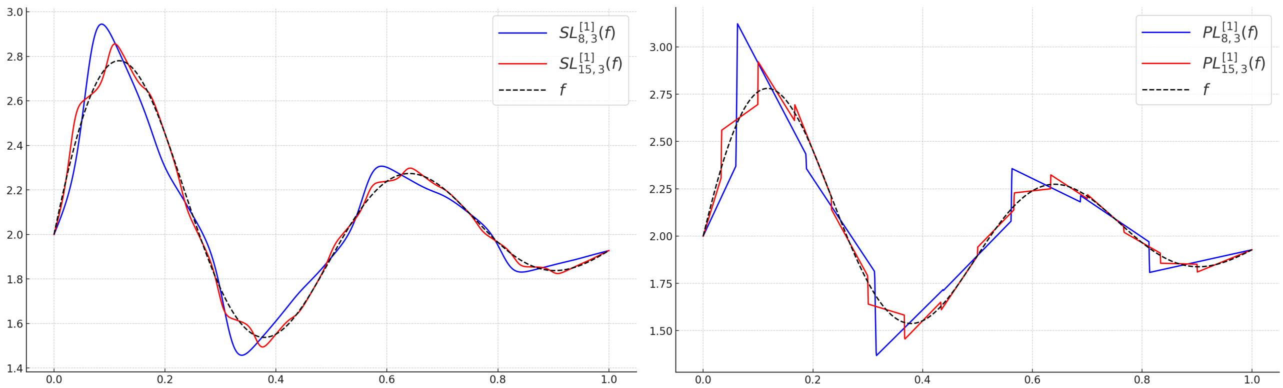

Figure 3: Approximation by the Shepard-Lagrange operators (left) and the max-product Shepard-Lagrange operators (right)

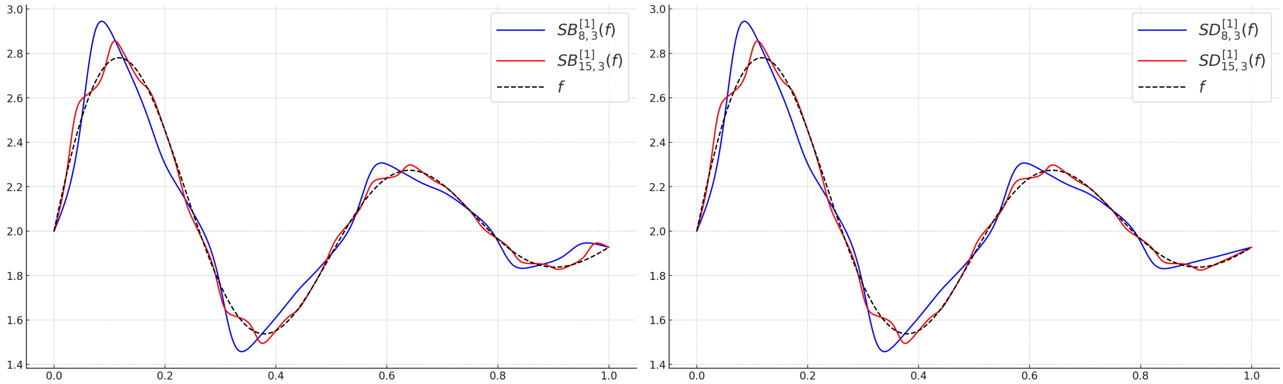

Figure 4: Approximation by the Shepard-Bernoulli operators (left) and the Shepard-Lidstone operators (right)

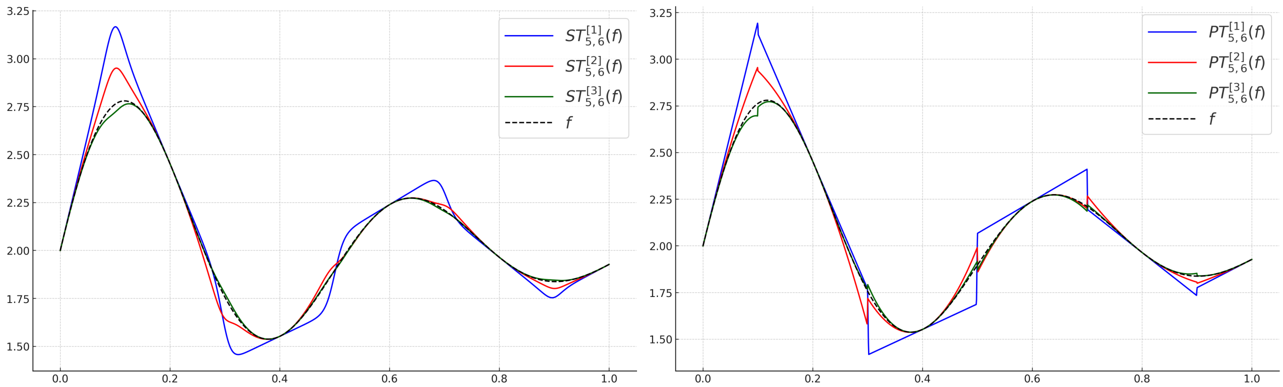

Figure 5: Approximation by the Shepard-Taylor operators (left) and the max-product Shepard-Taylor operators (right) for increasing values of r

We now present a comparative table of the approximation errors for the hybrid Shepard-type operators. We emphasize that the numerical values in this table were obtained in [43]. The comparison is carried out in terms of the mean square error, defined as

$$e_{ms}^{[r]} := \sqrt{\frac{1}{n_e} \sum_{j=1}^{n_e} \left(e_j^{[r]} \right)^2},$$

where $e_j^{[r]}$ denotes the absolute error

$$e_j^{[r]} := \left| SL_{n,2}^{[r]}(g) - g \right|$$

computed at $n_e = 100$ equally spaced points in the interval $[0, 1]$. In these computations, we use the test function $g(x) = \sin(2\pi x)$ with $r = 1$, $\lambda = 2$, and the values $n = 20, 40, 60$.

n	$ST_{n,\lambda}^{[r]}$	$SL_{n,\lambda}^{[r]}$	$SB_{n,\lambda}^{[r]}$
20	0.00025331	0.00023586	0.00974248
40	0.00006230	0.00006127	0.00275313
60	0.00002746	0.00002726	0.00127623

Table 1: Mean squared error comparison for Shepard-Taylor, Shepard-Lagrange, and Shepard-Bernoulli operators

7 Concluding Remarks

In this survey, we have examined the main techniques developed in the literature to enhance the approximation ability of the classical Shepard operators. We have focused specifically on hybrid modifications such as the Shepard-Taylor, Shepard-Lagrange, Shepard–Bernoulli, and Shepard-Lidstone operators. We have summarized their definitions and approximation properties, and discussed their advantages over the classical operators.

We note that both linear and nonlinear settings can benefit from such constructions. We also expect further refinements to appear in the near future. In particular, aspects such as computational complexity and cost should also be taken into account.

Our aim was to give a unified view of the developments in this area. We hope that this survey can serve as a reference for researchers and may guide future work on extending these ideas, adapting them to multivariate settings, or linking them with modern computational methods.

References

[1] D. Shepard, A two-dimensional interpolation function for irregularly-spaced data. Proceedings of the 23rd ACM National Conference, New York: ACM Press: 517–524, 1968. DOI: 10.1145/800186.810616

[2] M. Jing and J. Wu, Fast image interpolation using directional inverse distance weighting for real-time applications. Optics Communications, 286 (2013), 111–116. DOI: 10.1016/j.optcom.2012.09.011

[3] W. De Mulder, G. Molenberghs and G. Verbeke, A generalization of inverse distance weighting and an equivalence relationship to noise-free Gaussian process interpolation via Riesz representation theorem, Linear Multilinear Algebra 66 (2018), no. 5, 1054–1066. DOI: 10.1080/03081087.2017.1337057

[4] C. van Mierlo, M. G. R. Faes and D. Moens, Inhomogeneous interval fields based on scaled inverse distance weighting interpolation. Comput. Methods Appl. Mech. Engrg. 373 (2021), Paper No. 113542, 18 pp. DOI: 10.1016/j.cma.2020.113542

[5] J. Szabados, Direct and converse approximation theorems for the Shepard operator. Approx. Theory Appl. 7 (1991), no. 3, 63–76. DOI: 10.1007/BF02836457

[6] B. Della Vecchia and G. Mastroianni, Pointwise simultaneous approximation by rational operators. J. Approx. Theory 65 (1991), no. 2, 140–150. DOI: 10.1016/0021-9045(91)90099-V

[7] G. Criscuolo and G. Mastroianni, Estimates of the Shepard interpolatory procedure. Acta Math. Hungar. 61 (1993), no. 1-2, 79–91. DOI: 10.1007/BF01872100

[8] T. Hermann, Rational interpolation of periodic functions. Rend. Circ. Mat. Palermo (2) Suppl. No. 33 (1993), 337–344.

[9] B. Della Vecchia, Direct and converse results by rational operators. Constr. Approx. 12 (1996), no. 2, 271–285. DOI: 10.1007/BF02433043

[10] U. Amato and B. Della Vecchia, Weighting Shepard-type operators. Comput. Appl. Math. 36 (2017), no. 2, 885–902. DOI: 10.1007/s40314-015-0263-y

[11] B. Bede, H. Nobuhara, M. Daňková, A. Di Nola, Approximation by pseudo-linear operators, Fuzzy Sets Syst. 159 (7) (2008) 804–820. DOI: 10.1016/j.fss.2007.11.007

[12] X. Zhou, The saturation class of Shepard operators. Acta Math. Hungar. 80 (1998), no. 4, 293–310. DOI: 10.1023/A:1006538323418

[13] D. S. Yu and S. Zhou, Approximation by rational operators in L^p spaces. Math. Nachr. 282 (2009), no. 11, 1600–1618. DOI: 10.1002/mana.200610812

[14] D. S. Yu, On weighted approximation by rational operators for functions with singularities. Acta Math. Hungar. 136 (2012), no. 1–2, 56–75. DOI: 10.1007/s10474-011-0187-y

[15] D. Yu, On approximation by max-product Shepard operators. Results Math. 77 (2022), no. 6, Paper No. 219, 14 pp. DOI: 10.1007/s00025-022-01746-w

[16] G. Wang and D. Yu, Approximation by radial Shepard operators on scattered data. Anal. Math. Phys. 12 (2022), no. 5, Paper No. 127, 12 pp. DOI: 10.1007/s13324-022-00746-x

[17] R. Cavoretto, A. De Rossi, F. Dell'Accio and F. Di Tommaso, Fast computation of triangular Shepard interpolants. J. Comput. Appl. Math. 354 (2019), 457–470. DOI: 10.1016/j.cam.2018.03.012

[18] R. Cavoretto, A. De Rossi, F. Dell'Accio and F. Di Tommaso, An efficient trivariate algorithm for tetrahedral Shepard interpolation. J. Sci. Comput. 82 (2020), no. 3, Paper No. 57, 15 pp. DOI: 10.1007/s10915-020-01159-3

[19] F. Dell'Accio, F. Di Tommaso and K. Hormann, On the approximation order of triangular Shepard interpolation. IMA J. Numer. Anal. 36 (2016), no. 1, 359–379. DOI: 10.1093/imanum/dru065

[20] F. Dell'Accio, F. Di Tommaso, G. Ala and E. Francomano, Electric scalar potential estimations for non-invasive brain activity detection through multinode Shepard method MELECON 2022 - IEEE Mediterranean Electrotechnical Conference, Proceedings, 2022, pp. 1264–1268. DOI: 10.1109/MELECON53508.2022.9842881

[21] F. Dell'Accio, F. Di Tommaso, O. Nouisser and N. Siar, Solving Poisson equation with Dirichlet conditions through multinode Shepard operators. Comput. Math. Appl. 98 (2021), 254–260. DOI: 10.1016/j.camwa.2021.07.021

[22] T. Y. Gokcer and O. Duman, Approximation by max-min operators: a general theory and its applications. Fuzzy Sets and Systems 394 (2020), 146–161. DOI: 10.1016/j.fss.2019.11.007

[23] T. Y. Gokcer and O. Duman, Regular summability methods in the approximation by max-min operators. Fuzzy Sets and Systems 426 (2022), 106–120. DOI: 10.1016/j.fss.2021.03.003

[24] O. Duman and B. Della Vecchia, Complex Shepard operators and their summability. Results Math. 76 (2021), no. 4, Paper No. 214. DOI: 10.1007/s00025-021-01520-4

[25] O. Duman and B. Della Vecchia, Approximation to integrable functions by modified complex Shepard operators. J. Math. Anal. Appl. 512 (2022), no. 2, Paper No. 126161, 13 pp. DOI: 10.1016/j.jmaa.2022.126161

[26] O. Duman and B. Della Vecchia, Vector-valued Shepard processes: Approximation with summability, Axioms, 12 (2023), Paper No. 1124, 16 pp. DOI: 10.3390/axioms12121124

[27] O. Duman, B. Della Vecchia and E. Erkus-Duman, Kantorovich version of vector-valued Shepard operators, Axioms 13 (2024), Paper No. 181, 12 pp. DOI: 10.3390/axioms13030181

[28] O. Duman and E. Erkus-Duman, Nonlinear approximation of vector-valued functions by Shepard operators based on max-product and max-min operations. Fuzzy Sets and Systems 509 (2025), Paper No. 109332, 14 pp. DOI: 10.1016/j.fss.2025.109332

[29] R. Farwig, Rate of convergence of Shepard's global interpolation formula. Math. Comp. 46 (1986), no. 174, 577–590.

[30] G. Coman, Hermite-type Shepard operators. Rev. Anal. Numér. Théor. Approx. 26 (1997), no. 1-2, 33–38.

[31] G. Coman and L. Tâmbulea, A Shepard-Taylor approximation formula. Studia Univ. Babeş-Bolyai Math. 33 (1988), no. 3, 65–73.

[32] G. Coman and R. T. Trîmbiţăş, Combined Shepard univariate operators, East J. Approx. 7 (2001), 471–483.

[33] R. T. Trîmbiţăş, Univariate Shepard-Lagrange interpolation. Kragujevac J. Math. 24 (2002), 85–94.

[34] T. Cătinaş, The combined Shepard-Lidstone bivariate operator. Trends and applications in constructive approximation, 77–89, Internat. Ser. Numer. Math., 151, Birkhäuser, Basel, 2005. DOI: 10.1007/3-7643-7356-3_7

[35] T. Cătinaş, The bivariate Shepard operator of Bernoulli type. Calcolo 44 (2007), no. 4, 189–202. DOI: 10.10092-007-0136-x

[36] R. Caira and F. Dell'Accio, Shepard-Bernoulli operators. Math. Comp. 76 (2007), no. 257, 299–321.

[37] D. S. Yu, Approximation by Shepard-Lagrange operators. Acta Math. Sinica (Chinese Ser.) 53 (2010), no. 1, 97–108. DOI: 10.12386/A2010sxxb0014

[38] R. Caira, F. Dell'Accio and F. Di Tommaso, On the bivariate Shepard-Lidstone operators. J. Comput. Appl. Math. 236 (2012), no. 7, 1691–1707. DOI: 10.1016/j.cam.2011.10.001

[39] F. A. Costabile, F. Dell'Accio and F. Di Tommaso, Enhancing the approximation order of local Shepard operators by Hermite polynomials. Comput. Math. Appl. 64 (2012), no. 11, 3641–3655. DOI: 10.1016/j.camwa.2012.10.004

[40] F. Dell'Accio and F. Di Tommaso, Complete Hermite-Birkhoff interpolation on scattered data by combined Shepard operators. J. Comput. Appl. Math. 300 (2016), 192–206. DOI: 10.1016/j.cam.2015.12.016

[41] F. Dell'Accio and F. Di Tommaso, Bivariate Shepard-Bernoulli operators. Math. Comput. Simulation 141 (2017), 65–82. DOI: 10.1016/j.matcom.2017.07.002

[42] O. Duman, Max-product Shepard operators based on multivariable Taylor polynomials, J. Comput. Appl. Math. 437 (2024), Paper No. 115456, 15 pp. DOI: 10.1016/j.cam.2023.115456

[43] O. Duman, Improved approximation via hybrid Shepard-Lagrange operators: Linear and nonlinear perspectives, J. Comput. Appl. Math. (*to appear*). DOI: 10.1016/j.cam.2025.116864

[44] R. E. Barnhill, R. P. Dube and F. F. Little, Properties of Shepard's surfaces. Rocky Mountain J.

Math. 13 (1983), no. 2, 365–382. DOI: 10.1216/RMJ-1983-13-2-365
[45] R. E. Barnhill and H. S. Ou, Surfaces defined on surfaces. Comput. Aided Geom. Design 7 (1990), no. 1-4, 323–336. DOI: 10.1016/0167-8396(90)90040-X
[46] W. Maleika, Inverse distance weighting method optimization in the process of digital terrain model creation based on data collected from a multibeam echosounder. Applied Geomatics (2020) 12, no. 4, 397–407. DOI: 10.1007/s12518-020-00307-6
[47] Z. Li, X. Zhang, R. Zhu, Z. Zhang and Z. Weng, Integrating data-to-data correlation into inverse distance weighting. Comput. Geosci. 24 (2020), no. 1, 203–216. DOI: 10.1007/s10596-019-09913-9
[48] F. Dell'Accio and F. Di Tommaso, Scattered data interpolation by Shepard's like methods: classical results and recent advances. Dolomites Res. Notes Approx. 9 (2016), Special Issue, 32–44. DOI: 10.14658/PUPJ-DRNA-2016-Special_Issue-5
[49] F. Dell'Accio and F. Di Tommaso, Rate of convergence of multinode Shepard operators. Dolomites Res. Notes Approx. 12 (2019), 1–6. DOI: 10.14658/PUPJ-DRNA-2019-1-1
[50] P. J. Davis, Interpolation and Approximation. Dover Publications, New York, 1975.
[51] G. J. Lidstone, Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types, Proc. Edinb. Math. Soc. 2 (1929), 16–19. DOI: 10.1017/S0013091500007501
[52] H. Poritsky, On certain polynomial and other approximations to analytic functions, Trans. Amer. Math. Soc. 34 (1932), 274–331. DOI: 10.1073/pnas.16.1.83
[53] R. P. Agarwal and P. J. Y. Wong, Error Inequalities in Polynomial Interpolation and Their Applications. Kluwer Academic Publishers, Dordrecht, 1993.