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Survey Article

Bases and Isomorphisms of Whitney Spaces

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Dedicated to the memory of V. P. Zaharjuta

Abstract: We consider results related to bases and isomorphisms of Whitney spaces $\mathcal{E}(K)$ and the extension property of compact sets K. The first two sections discuss the notion of a topological basis and its importance in analysis. Then we consider model spaces of infinitely differentiable functions that may occur in applications. Our main interest is in Whitney spaces $\mathcal{E}(K)$ and the extension property of compact sets, that is, the existence of a continuous linear extension operator from $\mathcal{E}(K)$ to the space of infinitely differentiable functions on the whole Euclidean space. In Section 5 we consider what we believe to be the main methods for constructing such an operator. Section 6 contains some geometric and other conditions characterizing the extension property. Sections 7–11 are devoted to bases in spaces of infinitely differentiable functions, Whitney spaces, and restriction spaces, with an emphasis on the author's results obtained using the method of local interpolations. The final sections present results related to the isomorphic classification of these spaces. We consider the counting linear topological invariants (mainly the diametrical dimension), interpolation invariants (mainly generalizations of the dominated norm property), and compound invariants, which reduce to the computation of diametrical dimension for so-called synthetic neighborhoods. Various families of the continuum cardinality of pairwise non-isomorphic spaces are presented. Finally, some open problems are proposed. The review contains many examples, both classic and new.

Keywords: Topological bases; Whitney spaces; extension property; isomorphism

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1 Bases in Topological Vector Spaces

Let $\mathcal{X}$ be a topological vector space over the field $\mathbb{K}$ with topological dual $\mathcal{X}'$. A sequence $(e_n)_{n=1}^{\infty} \subset \mathcal{X}$ is a (topological) basis in $\mathcal{X}$ if for each $x \in \mathcal{X}$ there is a unique sequence $(\xi_n(x))_{n=1}^{\infty} \subset \mathbb{K}$ such that the series

$$\sum_{n=1}^{\infty} \xi_n(x)\, e_n \tag{1.1}$$

converges to x in the topology of $\mathcal{X}$. Therefore $(\xi_n)_{n=1}^{\infty}$ is a sequence of linear functionals. The system $(e_n, \xi_n)_{n=1}^{\infty}$ is biorthogonal which means $\xi_n(e_m) = \delta_{mn}$. The basis $(e_n)_{n=1}^{\infty}$ is called a *Schauder basis* if all functionals ξ_n are continuous.

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Suppose $(\mathcal{X}, ||\cdot||)$ is a Banach (B) space. Since the sequence of partial sums for the series (1.1) is bounded, the norm $|||x||| = \sup_N ||\sum_{n=1}^N \xi_n(f)\, e_n||$ is well defined on $\mathcal{X}$. In addition, $\mathcal{X}$ is complete in $|||\cdot|||$. Clearly, $||\cdot|| \le |||\cdot|||$. By the Open Mapping Theorem, the norms are equivalent. Hence there is a constant C with $|||x||| \le C\,||x||$ for $x \in \mathcal{X}$. The infimum of such constants is called the *basis constant* of $(e_n)_{n=1}^\infty$. It follows easily that the norms of basis projections $P_N : x \mapsto \sum_{n=1}^N \xi_n(x)\, e_n$, $N \in \mathbb{N}$ are bounded by C and $|\xi_n(x)| \le 2C||x||$, so $\xi_n \in \mathcal{X}'$. Similar argument can be applied to Fréchet (F) spaces. Thus, in the case of F-spaces, each topological basis is a Schauder basis. For Schauder bases in classical Banach spaces, see, for instance [77], 4.1 or [97], 1.2.

Example 1.1. *Monomials $(t^n)_{n=0}^\infty$ in the space $\mathcal{P}$ of all polynomials with the uniform norm $||P|| = \sup_{-1\le t\le 1}|P(t)|$ give an example of non-Schauder topological basis. Of course, $(t^n)_{n=0}^\infty$ is a basis in $\mathcal{P}$ with the biorthonormal functionals $\xi_n(P) = P^{(n)}(0)/n!$ However, ξ_1 is not continuous. Indeed, let T_k be the k-th Chebyshev polynomial of the first kind. See Section 7 and [91] for the definition and properties of these polynomials. Since $||T_k|| = 1$ for all k but $|T_k'(0)| = k$ for odd k it follows that the functional ξ_1 is not bounded.*

Typical spaces in analysis are either Banach or nuclear Fréchet (NF) spaces, see for instance Section 28 in [78] for the definition and properties of nuclear spaces. There are many different types of bases in B-spaces: conditional and unconditional (depending on the nature of converges of (1.1)), shrinking and boundedly complete, weak bases, etc. In particular, a basis $(e_n)_{n=1}^\infty$ is *absolute* if the series (1.1) converges absolutely. This nice property of bases is exceptional for B-spaces: if B-space $\mathcal{X}$ has an absolute basis then $\mathcal{X} \simeq l_1$, see, e.g. [63], p. 974. Here and throughout, $\simeq$ means an isomorphism, that is a linear homeomorphism.

On the other hand, by the significant Dynin-Mityagin theorem ([79], T.9), for NF-spaces the absoluteness of bases is mandatory. It should be noted that the absoluteness of bases in $A(\mathbb{D})$ was previously proved by Dragilev in [22].

Let the topology of NF-space $\mathcal{X}$ be given by semi-norms $(||\cdot||_p)_{p=1}^\infty$. In what follows, we can assume that $||\cdot||_p \le ||\cdot||_{p+1}$ for each p. If $(e_n)_{n=1}^\infty$ is a basis in $\mathcal{X}$ then for each $p \in \mathbb{N}$ and $x \in \mathcal{X}$, the series $\sum_{n=1}^\infty |\xi_n(x)| \cdot ||e_n||_p$ converges. Moreover, Theorem 9 in [79] suggests the following method to show the basic property of a given sequence $(e_n)_{n=1}^\infty$ with biorthogonal functionals $(\xi_n)_{n=1}^\infty$. Suppose that the functionals $(\xi_n)_{n=1}^\infty$ are total over $\mathcal{X}$, ($\xi_n(f) = 0$ for all $n \in \mathbb{N}$ implies $f = 0$), since otherwise, the linear span of $(e_n)_{n=1}^\infty$ is not dense in $\mathcal{X}$. If for each $p \in \mathbb{N}$ there are $q \in \mathbb{N}$ and $C > 0$ so that

$$|\xi_n|_{-q} \cdot ||e_n||_p \le C \quad \text{for all} \quad n, \tag{1.2}$$

then $(e_n)_{n=1}^\infty$ is the absolute basis for $\mathcal{X}$. Here $|\cdot|_{-q}$ denotes the dual norm: for $\xi \in \mathcal{X}'$ let $|\xi|_{-q} = \sup\{|\xi(x)| : ||x||_q \le 1\}$. Since it can take an infinite value, we adhere to the conventions that $0\cdot\infty = 0$. The following, despite its triviality, gives the first example of a basis in Whitney space in our survey.

Example 1.2. *The space of all real sequences ω is isomorphic to the Whitney space $\mathcal{E}(\{a\})$, where $\{a\}$ is a singleton, see Section 4. The product topology of ω can be defined by the semi-norms $||x||_p = \max\{|x_k| : 1 \le k \le p\}, p \in \mathbb{N}$, for $x = (x_k)_{k=1}^\infty \in \omega$. Clearly, the sequence of unit vectors $(e_n)_{n=1}^\infty$ with $e_n = (\delta_{nk})_{k=1}^\infty$ is a basis in ω. The dual of ω is the space ψ of finite sequences, see e.g. [71], p.288. The functionals $\xi_m : x \mapsto x_m, m \in \mathbb{N}$, are biorthogonal to $(e_n)_{n=1}^\infty$. Here $||e_n||_p = 1$ for $n \le p$ and $||e_n||_p = 0$ for $n \ge p+1$. In turn, $|\xi_n|_{-q} = 1$ for $n \le q$ and $|\xi_n|_{-q} = \infty$ for $n \ge q+1$. Thus (1.2) is valid with $q = p$ and $C = 1$.*

Example 1.3. *The B-space $C[0,1]$ with the uniform norm fails to have an unconditional basis, see, e.g. [97], T.15.1. Let $C(X)$ be the F-space of continuous functions on a completely regular space X. We equip it with the compact-open topology (of uniform convergence on compact subsets of X). Attempts to obtain an absolute basis in $C(X)$ reduce it to nuclear space. By [18], if $C(X)$ is complete, barreled and has an absolute basis, then $C(X) \simeq \omega$.*

2 Why are Bases so Important?

Knowing the basis, even the fact that the basis exists, is useful in the analysis of space. Firstly, the partial sums of (1.1) with rational coefficients are dense in the corresponding space, so the space with a basis is separable. In turn, separability implies other useful properties of space. For example, if $\mathcal{X}$ is a separable $B-$space then the closed unit ball of $\mathcal{X}'$ is metrizable.

Secondly, every Banach space $\mathcal{X}$ with a basis has the bounded approximation property, which means that each compact operator in $\mathcal{X}$ can be approximated (in the operator norm) by a bounded sequence of finite-rank operators, see, e.g. [77], T. 4.1.33.

Referring to [82], p.217, "Knowledge of bases made a number of results easier to prove and also gave a deeper insight into the properties of such spaces."

If we know a basis $(e_n)_{n=1}^{\infty}$ of a space $\mathcal{X}$ then instead of elements of $\mathcal{X}$ we can consider sequences of coefficients of their basic expansions. The homeomorphism $x \mapsto (\xi_n(x))_{n=1}^{\infty}$ maps $\mathcal{X}$ onto the *coordinate space*. Especially this is useful for NF-spaces, where due to absoluteness of bases, $\mathcal{X}$ is isomorphic to a Köthe space. Recall that a matrix $A = (a_{k,p})_{k,p=1}^{\infty}$ with non-negative entries is a Köthe matrix if $a_{k,p} \le a_{k,p+1}$ for all $k, p \in \mathbb{Z}_+$ and for each k there exists a p with $a_{k,p} > 0$. The corresponding Köthe space is defined as

$$\mathcal{K}(A) = \{x = (x_k)_{k=1}^{\infty} \in \omega : ||x||_p := \sum_{k=1}^{\infty} |x_k| \cdot a_{k,p} < \infty \text{ for all } p\}.$$

Thus $\mathcal{K}(A)$ is a projective limit of weighted l_1 spaces. The matrix $(k^p)_{k,p=1}^{\infty}$ determines the space of rapidly decreasing sequences

$$s = \{(x_k)_{k=1}^{\infty} : x_k \cdot k^p \to 0 \text{ as } k \to \infty \text{ for all } p \in \mathbb{N}\},$$

which plays a central role in the theory of $NF-$spaces and especially in our survey, since many spaces of infinitely differentiable functions are isomorphic to s.

Suppose that a $NF-$space $\mathcal{X}$ has a basis $(e_n)_{n=1}^{\infty}$. It is a simple matter to show that $\mathcal{X} \simeq \mathcal{K}(A)$ with $A = (||e_k||_p)_{k,p=1}^{\infty}$. Now it is easier to handle with the coordinate space than with the original space. In particular, the coordinate forms of linear topological invariants (Sections 12–14) are preferable.

The most important such invariant for us is the (DN) property. Due to Tidten (2.4 in [100], see also Th. 30.1 in [78]), given compact set $K \subset \mathbb{R}^N$ has the extension property (Section 4) if and only if the space $\mathcal{E}(K)$ has a dominating norm.

Recall that a norm $|| \cdot ||_p$ is dominating in a Fréchet space if for each $q \ge p$ there is $r \ge q$ and $C > 0$ with

$$|| \cdot ||_q^2 \le C || \cdot ||_p \, || \cdot ||_r. \tag{2.1}$$

If we know a basis in $\mathcal{E}(K)$ then it is enough to check (2.1) only for the elements of basis. From the above it follows that bases are of great importance for the isomorphic classification of spaces. Below we present some results related to such a classification of $C^{\infty}-$function spaces and Whitney spaces.

Knowledge of a basis in a linear topological space allows one to consider matrix forms of automorphisms of this space.

Another important application of bases is related to extension operators. For example, in the theory of differential equations, to use Fourier methods, it is necessary to extend a given function from a given region to the entire space, preserving the type of smoothness of the function, see, e.g. [62], 2.3. Of course, the extension given by a linear bounded operator is preferable. If for a space with a basis one can specify suitable individual extensions of the basis elements, then the extension operator can be defined by linearity. See Section 5 for more details.

It should also be noted that functions which form a basis in the functional space $\mathcal{X}$ usually play a special role in concrete problems of analysis related to $\mathcal{X}$. We can mention here the Chebyshev polynomials in the space $C^\infty[-1,1]$([79]), the Hermite functions in the space S of rapidly decreasing $C^\infty-$functions ([79]), the Faber polynomials in spaces of analytic functions (see, e.g. [99]) and the Franklin sequence in the Hardy space H^1 ([120]). Wavelets, widely used for diverse scientific applications, form unconditional bases in a variety of functional spaces on $\mathbb{R}^N$ (see, e.g. [60] and [121]).

Due to the importance of bases in analysis and applications, many questions have been raised in the literature about the existence or properties of bases. Let us mention two famous "existence problems" for Banach and $NF-$ spaces. Both were solved negatively in the mid-70s.

1) "It is not known if every separable Banach space has a base" was posed in [9], p.68.

The first counterexample was given by Enflo in [27] who presented a separable Banach space without the approximation property. In a short time, many simplifications have been proposed.

2) At the end of his memoir, Grothendieck proposed several open problems. He first discussed the bounded approximation problem and regarded it ([58], Ch.2, p.135) as "probably the most important problem encountered". In [128], Zobin and Mityagin found the first example of $NF-$space without basis. Later, other approaches were suggested in [83] by Moscatelli and in [26], where Dubinsky constructed an $NF-$space without the bounded approximation property. Vogt showed that spaces of this kind can be given using C^∞-functions. The space in [113] has no basis, while in [114] it does not even have the property of bounded approximation. The spaces were equipped with special topologies.

3 Spaces $C^\infty(\Omega)$ and $C^\infty(\bar{\Omega})$

In the late 1940s, Laurent Schwartz proposed a revolutionary idea for generalizing the concept of a function. His approach made it possible to apply Fourier transform method to many problems in partial differential equations where traditional methods no longer worked. In [94] he introduced generalized functions as linear functionals over spaces of test functions. This has significantly increased the interest in C^∞-functions, since test functions are exactly that.

Let us consider model spaces of C^∞-functions over the field $\mathbb{K}$. Suppose $\Omega \neq \emptyset$ is an open subset of $\mathbb{R}^N$. For a given multi-index $\alpha = (\alpha_j)_{j=1}^N \in \mathbb{Z}_+^N$, let $|\alpha| = \sum_{j=1}^N \alpha_j$, $\alpha! = \prod_{j=1}^N \alpha_j!$ and D^α be the differentiation operator $(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$. A function $f : \Omega \longrightarrow \mathbb{K}$ is said to belong to $C^\infty(\Omega)$ if $D^\alpha f$ (we will also use the notation $f^{(\alpha)}$) is continuous on Ω for each α. The natural compact-open topology τ_c of $C^\infty(\Omega)$ is given by the semi-norms

$$|f|_{p,K} := \sup\{|D^\alpha f(x)| : x \in K, |\alpha| \le p\}, \tag{3.1}$$

where $p \in \mathbb{Z}_+$ and K is a compact subset of Ω. Since Ω can be represented as a countable union of compact sets, the space $C^\infty(\Omega)$ is metrizable.

For a given compact subset K of Ω, let $\mathcal{D}_K$ denote the space of functions from $C^\infty(\Omega)$ whose supports lies in K and $\mathcal{D}(\Omega)$ be the union of $\mathcal{D}_K$ for all compact $K \subset \Omega$. The above semi-norms $|\cdot|_{p,K}$ are well defined on $\mathcal{D}(\Omega)$, but the space is not complete in τ_c, see, e.g. [92], p.137. For this reason, the space $\mathcal{D}(\Omega)$ is considered as $LF-$space, that is a strict inductive limit of Fréchet spaces, see, e.g. [71], 19.2. The corresponding topology is locally convex, complete and has many useful properties, except for metrizability. The notation $\mathcal{D}$ for $\mathcal{D}(\mathbb{R}^N)$ goes back to Schwartz, while in other books the authors use the symbol $C_c^\infty(\Omega)$ ([104]) or $C_0^\infty(\Omega)$ ([107]) instead of $\mathcal{D}(\Omega)$. Distributions or generalized functions are just elements of the strong dual space $\mathcal{D}'(\Omega)$.

The next important space of test functions is the Schwartz space S of rapidly decreasing at infinity functions on $\mathbb{R}^N$. Elements of its dual S' are tempered distributions. The Fourier transform is an automorphism of S and, by duality, also an automorphism of S'. The space S consists of all functions from $C^\infty(\mathbb{R}^N)$, for which

$$||f||_{p,m} := \sup_{|\alpha|\le p} \sup_{x\in\mathbb{R}^N} (1+|x|)^m |D^\alpha f(x)| \tag{3.2}$$

is finite for all $p, m \in \mathbb{Z}_+$. By Lemma 27 in [79], in the one-dimensional case, the Hermite functions form a basis in the space S and it is isomorphic to s. Later this isomorphism was shown for each N.

These spaces can be generalized to the case of bounded domains Ω in $\mathbb{R}^N$. Let $\rho \in C^\infty(\Omega)$ be a positive function that increases to infinity as its argument goes to the boundary of Ω. If ρ satisfies some mild conditions, see [107], 6.2.1, then the generalization of norms (3.2) (instead of $1+|x|$ we put $\rho(x)$) define the F-space $S_\rho(\Omega)$ of functions rapidly decreasing at the boundary of Ω. Following Triebel ([107]), by $\bar{C}_0^\infty(\Omega)$ we denote the space of $C^\infty-$ functions which support belongs to $\bar{\Omega}$. The $F-$topology of $\bar{C}_0^\infty(\Omega)$ is given by the norms $||f||_p := \sup_{|\alpha|\le p} \sup_{x\in\Omega} |D^\alpha f(x)|, p \in \mathbb{Z}_+$. This space is isomorphic to $S_\rho(\Omega)$ for regular it a certain sense ρ . By the section 8.3.2 in [107], $S_\rho(\Omega) \simeq s$ for arbitrary domain Ω in $\mathbb{R}^N$ and ρ with $\rho^{-a} \in L_1(\Omega)$ for some $a \ge 0$. Also $\bar{C}_0^\infty(\Omega) \simeq s$ for each bounded domain Ω . Thus, for isomorphism of these spaces with s no restrictions on the smoothness of $\partial\Omega$ are required.

One more class of infinitely differentiable functions is formed by the spaces $C^\infty(\bar{\Omega})$ for a bounded open set Ω. There are different definitions of this space. Usually ([104], p.99, or [78], p.39 (6), for finite degree) $C^\infty(\bar{\Omega})$ consists of functions f from $C^\infty(\Omega)$ that together with all derivatives are uniformly continuous on Ω. Thus every $D^\alpha f$ can be extended continuously to $\bar{\Omega}$. In [107], this space is denoted as $\bar{C}^\infty(\Omega)$. If $\Omega = (a, b)$ and $p \le \infty$ then $C^p[a, b]$ denotes $C^p([a, b])$.

An $F-$topology of $C^\infty(\bar{\Omega})$ is given by (3.1) with $K = \bar{\Omega}, p \in \mathbb{Z}_+$, now they are norms. However, in some cases, the corresponding space is not nuclear, not even Montel. Recall that an F-space is Montel if each of its bounded sets is precompact. One can take a bounded domain of comb-type with the infinite number of “teeth”, see, for example [76], p. 145. By means of a smoothing function, it is easy to present a bounded sequence of functions $(f_n)_{n=1}^\infty \subset C^\infty(\bar{\Omega})$ with the same norm $|\cdot|_0$ and support of f_n belonging to $n-$th “tooth”. The sequence is not precompact, so the space $C^\infty(\bar{\Omega})$ is not Montel.

Let’s consider a one-dimensional analogue of such an example.

Example 3.1. *Let $\Omega = [0, 1] \setminus K_0$, where K_0 is the classical Cantor ternary set. Then the Cantor staircase function f is uniformly continuous and f' is identically zero on Ω. Thus $f \in C^\infty(\bar{\Omega})$. Similar to the construction of f, we can take uniformly continuous functions f_n with zero derivatives, such that the supports of functions are disjoint, $|f_n|_0 = 1$ and $f_n = 1$ on the interval $[3^{-n}, 2\cdot 3^{-n}]$ for $n \in \mathbb{N}$. The sequence $(f_n)_{n=1}^\infty$ is bounded but not precompact so $C^\infty(\bar{\Omega})$ is not Montel. Here $C^\infty(\bar{\Omega})$ is not isomorphic to $C^\infty[0, 1]$, even though $\bar{\Omega} = [0, 1]$.*

To avoid such situations, some authors ([72], 3.3.7, [76], 1.1) define $C^p(\bar{\Omega}), p \le \infty$, as the space of restrictions on $\bar{\Omega}$ functions from $C^p(\mathbb{R}^N)$, which gives us precisely the Whitney space $\mathcal{E}^p(K)$ for $K = \bar{\Omega}$ that will be considered in the next section.

Triebel offers the usual definition for $C^\infty(\bar{\Omega})$, but for domains with a sufficiently smooth boundary: smooth in [106] and Lipschitz in [108], so here again $C^\infty(\bar{\Omega}) \simeq \mathcal{E}(\bar{\Omega})$.

The difference between the definitions can also be illustrated by the following standard example of the inverse sharp cusp Ω and the non extendable function from $C^\infty(\bar{\Omega})$.

Example 3.2. *Let $\psi_0(x) = \exp(-1/x)$ for $0 < x < 1$ and*

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |y| < 1 \ \textit{for} \ -1 < x \le 0 \ \textit{and} \ \psi_0(x) < |y| < 1 \ \textit{for} \ 0 < x < 1\}.$$

For $(x,y)\in\Omega$ let $f(x,y)=0$ if $x\le 0$, so $f'_y(0,0)=1$, and $f(x,y)=\operatorname{sign} y\cdot\psi_0(x)$ if $x>0$. Then the function belongs to $C^\infty(\bar\Omega)$ in the sense of first definition, but f cannot be extended to $\tilde f\in C^\infty(\mathbb{R}^2)$ since, by the Mean Value Theorem, for each $x>0$ there is $\theta_x\in(-\psi_0(x),\psi_0(x))$ with $\tilde f'_y(x,\theta_x)=1$, so $\tilde f'_y(0,0)=1$, a contradiction.

The sharp cusp

$$\Omega_{\psi_0}=\{(x,y)\in\mathbb{R}^2: 0<x<1, |y|<\psi_0(x)\} \tag{3.3}$$

is also interesting for us domain because (2.1) is not valid for the space $C^\infty(\bar\Omega_{\psi_0})$. Hence it is not isomorphic to s. Moreover, families of continuum cardinality of pairwise non isomorphic spaces of this kind can be represented using linear topological invariants, see Section 13 for details.

The survey reflects research interests of the author, so we do not consider here spaces of ultra-differentiable function and some other classes of C^∞-functions.

4 Whitney Spaces

Suppose we are given a compact set $K\subset\mathbb{R}^N$ and $p\in\mathbb{Z}_+$. Then the product $\prod_{|\alpha|\le p}C(K)$ is a Banach space of jets $\mathbf{f}=(f_\alpha(x))_{x\in K,|\alpha|\le p}$ of continuous on K functions with the norm $||\mathbf{f}||_p:=\sup\{|f_\alpha(x)|: x\in K, |\alpha|\le p\}$. In some cases, the function $f=f_0$ determines all $f_\alpha, |\alpha|\le p$, as the corresponding derivatives $f^{(\alpha)}$ if they exist. This happens, for example, when K is a bounded perfect subset of $\mathbb{R}$ or $K=\overline{\Omega}$ for a bounded domain $\Omega\subset\mathbb{R}^d$. Then $C^p(K)$ is well-defined as a closed subspace of $\prod_{|\alpha|\le p}C(K)$.

Let B be a closed cube containing K. The Whitney space $\mathcal{E}^p(K)$ of extendable jets consist of all $\mathbf{f}\in\prod_{|\alpha|\le p}C(K)$ such that there exists $F\in C^p(B)$ with $F^{(\alpha)}|_K=f_\alpha$ for $|\alpha|\le p$. In this case we use the notation $f=(f^{(\alpha)}(x))_{x\in K,|\alpha|\le p}$ for such $\mathbf{f}$.

Thus the space $\mathcal{E}^p(K)$ is a factor space of $C^p(B)$ and it should be equipped with the quotient norm $|||\,f\,|||_p=\inf\ |\,F\,|_{p,B}$, where the infimum is taken over all possible extensions of f to $F\in C^p(B)$ and $|\cdot|_{p,B}$ is given in (3.1). Naturally, the Fréchet spaces $C^\infty(K)$ and $\mathcal{E}(K):=\mathcal{E}^\infty(K)$ are defined as the corresponding projective limits, τ_Q is the quotient topology of $\mathcal{E}(K)$.

For example, Borel proved that for each sequence $(a_j)_{j=0}^\infty$ there exists a function F from $C^\infty(\mathbb{R})$ with $F^{(j)}(0)=a_j$ for $j\in\mathbb{Z}_+$, see [105], T.18.1 for the multidimensional case. Hence $\omega=\mathbb{R}^{\mathbb{N}}$ as sets and $f=(a_j)_{j=0}^\infty\in\mathcal{E}(\{0\})$.

For each jet $\mathbf{f}=(f_\beta(x))_{x\in K,|\beta|\le p}$ from $\prod_{|\alpha|\le p}C(K)$ and $y\in K$, let

$$R_y^p\mathbf{f}(x)=f_0(x)-T_y^p\mathbf{f}(x)=f_0(x)-\sum_{|\beta|\le p}\frac{f_\beta(y)}{\beta!}(x-y)^\beta$$

be the formal Taylor remainder of order p for $\mathbf{f}$ at the point y. Its formal α-derivative (for $|\alpha|\le p$) is

$$(R_y^p\mathbf{f})^{(\alpha)}(x)=f_\alpha(x)-\sum_{|\beta|\le p-|\alpha|}\frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta.$$

If $\mathbf{f}=f\in\mathcal{E}^p(K)$, so $(R_y^pf)^{(\alpha)}(x)=(R_y^pF)^{(\alpha)}(x)$ for some $F\in C^p(B)$, then, by Peano's form of the remainder,

$$(R_y^pf)^{(\alpha)}(x)=o(|x-y|^{p-|\alpha|})\ \text{ for }\ x,y\in K\ \text{ and } |\alpha|\le p,\ \text{ as }\ |x-y|\to 0. \tag{4.1}$$

By Whitney's extension theorem ([118]), this condition is also sufficient for the jet $\mathbf{f}$ to be extendable. Moreover, an extension F can be constructed using a continuous linear mapping

$$W^p:\mathcal{E}^p(K)\longrightarrow C^p(B): f\mapsto F, \tag{4.2}$$

where F is the restriction to B of the following function $\tilde{F}$ from $C^p(\mathbb{R}^N)$. If $x \in K$ then $\tilde{F}(x) = f_0(x)$, otherwise

$$\tilde{F}(x) = \sum_j \psi_j(x)\, T^p_{a_j} f(x). \tag{4.3}$$

Here $(\psi_j)_{j=1}^{\infty}$ is a special partition of unity on $\mathbb{R}^N \setminus K$ with controllable supports S_j and derivatives of ψ_j; the point $a_j \in K$ realizes the distance from S_j to K, see [118] or T.2.3 in [14], T.2.3 in [75] for more details.

The characterization (4.1) of jets f from $\mathcal{E}^p(K)$ implies the following definition of a norm in this space

$$\| f \|_p = |f|_{p,K} + \sup \left\{ \frac{|(R^p_y f)^{(\alpha)}(x)|}{|x-y|^{p-|\alpha|}} \text{ with } x, y \in K, x \neq y, |\alpha| \leq p \right\}. \tag{4.4}$$

Since $(R^p_y f)^{(\alpha)}(x) = (R^{p-1}_y f)^{(\alpha)}(x) - \sum_{|\beta|=p-|\alpha|} \frac{f^{(\alpha+\beta)}(y)}{\beta!}(x-y)^\beta$, by Lagrange's form of the remainder of order $p-1$, there is a constant $c(p,N)$ so that $\| f \|_p \leq c(p,N)^{-1}|F|_{p,B}$ for any extension F. Hence the norm $\| \cdot \|_p$ is not weaker than $||| \cdot |||_p$ on $\mathcal{E}^p(K)$. The space is complete in both norms, so, by the Open Mapping Theorem, there exists $C(p,N)$ such that

$$c(p,N)\,||\, f\,||_p \leq |||\, f\,|||_p \leq C(p,N)\,||\, f\,||_p \quad \text{for} \quad f \in \mathcal{E}^p(K). \tag{4.5}$$

Thus, the norms $|| \cdot ||_p$ and $||| \cdot |||_p$ are equivalent and the quotient topology τ_Q coincides on $\mathcal{E}(K)$ with the Whitney topology τ_W given by (4.4) for $p \in \mathbb{Z}_+$, now they are semi-norms.

The condition (4.1) cannot be weakened to $(R^p_y f)^{(\alpha)}(x) = o(|x-y|^{p-|\alpha|})$ for $y \in K$ and $|\alpha| \leq p$, as $x \to y, x \in K$.

Example 4.1. ([118], *page* 65) *Let* $p = 1, K = \{0\} \bigcup (2^{-s})_{s=1}^{\infty} \bigcup (2^{-s} + 2^{-2s})_{s=1}^{\infty}, f(0) = f(2^{-s}) = 0, f(2^{-s} + 2^{-2s}) = 2^{-2s}$ *and* $f' \equiv 0$. *Then the above condition is valid for each fixed* y *but* f *cannot be extended to* $F \in C^1(\mathbb{R})$. *See also Remark* 2.4 *in* [14].

Furthermore, all α with $|\alpha| \leq p$ must be considered in (4.4). Otherwise, in general, the corresponding space is not complete. Let us consider the norm in $\mathcal{E}^p(K)$

$$\rceil\, f\, \lceil_p = |f|_{p,K} + \sup \left\{ \frac{|R^p_y f(x)|}{|x-y|^p} \text{ with } x, y \in K, x \neq y \right\}.$$

Example 4.2. *For the set* K *from the previous example, let* $a_s = 2^{-s}$, $b_s = 2^{-s} + \varepsilon_s$ *with* $\varepsilon_s = 2^{-2s}$. *Consider the jet* f_s *with all* $f_s^{(j)}(x) = 0$ *except* $f_s'(b_s) = \varepsilon_s^2, f_s''(b_s) = 2\varepsilon_s$, *so* f_s *is defined by function* $F_s(x) = \varepsilon_s(x-a_s)(x-b_s)$ *at* b_s. *Then all* $R^3_y f(x) = 0$ *except maybe* $R^3_{b_s} f(x)$. *In addition,* $R^p_{b_s} f(a_s) = 0$ *for* $p \geq 2$. *It follows that* $\rceil\, f_s \lceil_3 \to 0$ *as* $s \to \infty$, *as is easy to check. However,* $||f_s||_3 \geq \varepsilon_s^{-2}\,|(R^3_{b_s} f)'(a_s)| = 1$. *Therefore, the norms* $\rceil \cdot \lceil_3$ *and* $|| \cdot ||_3$ *are not equivalent and the space* $\mathcal{E}^3(K)$ *is not complete in* $\rceil \cdot \lceil_3$.

Let Ω be the domain from Example 3.2 and $K = \bar{\Omega}$. Then $\mathcal{E}(K)$ is a proper subspace of $C^\infty(\bar{\Omega})$. The following natural question arises: when is the norm (4.4) equivalent to $| \cdot |_{p,K}$? Of course, this means the possibility to estimate from above the second term on the right side of (4.4) by means of the first. This can be done for $p-$regular sets ([14] , 2.13), see also condition P in [119] and regularly situated sets in [75], 5. Recall that a compact set $K \subset \mathbb{R}^N$ is *p-regular* if for some constant $C > 0$ the condition $d_g(x,y)^p \leq C|x-y|$ is valid for $x, y \in K$. Here we assume that $K \subset \mathbb{R}^N$ is arc connected and $d_g(x,y)$ is the geodesic distance on K. We say that K is *Whitney regular* if it is p-regular for each $p \in \mathbb{N}$. Thus the above set is not Whitney regular.

Some methods of homological algebra can be successfully used in the structure theory of locally convex spaces. Short exact sequences are very convenient for working with subspaces and quotient spaces of a given space. See, for example, section 26 in [78] for the definition of short exact sequences and basic facts about their splitting.

Since $C^p(B)$ is an algebra, we can consider the ideal of $p-$flat functions on B for a given compact set $K \subset B:$

$$\mathcal{F}^p(K,B) = \{F \in C^p(B) : F^{(\alpha)}|_K = 0 \ \text{ for all } \ |\alpha| \le p\}.$$

The short sequence

$$0 \longrightarrow \mathcal{F}^p(K,B) \xrightarrow{i} C^p(B) \xrightarrow{\pi} \mathcal{E}^p(K) \longrightarrow 0 \tag{4.6}$$

is exact, since by Whitney's extension theorem, the operator π is surjective. The map (4.2) is a right inverse of π, so the sequence (4.6) splits for each compact set K and finite p. In the infinite case situation is different. Here for each $p \in \mathbb{Z}_+$ and fixed $f \in \mathcal{E}(K)$, one can find a function $h_p \in \mathcal{F}(K,B)$ with $|\, W^{p+1}(f) - W^p(f) - h_p\,|_{p,B} \le 2^{-p}$, see for example [75], T.4.1. Then the function $F = W_0(f) + \sum_{p=0}^{\infty}[W_{p+1}(f) - W_p(f) - h_p]$ is well-defined, extends f, and belongs to $C^\infty(B)$. Hence the short sequence

$$0 \longrightarrow \mathcal{F}(K,B) \xrightarrow{i} C^\infty(B) \xrightarrow{\pi} \mathcal{E}(K) \longrightarrow 0 \tag{4.7}$$

is also exact. Here, of course,

$$\mathcal{F}(K,B) = \{F \in C^\infty(B) : F^{(\alpha)}|_K = 0 \ \text{ for all } \ \alpha\}. \tag{4.8}$$

However, the method of constructing extensions described above is not linear. Using a Hamel basis of the space $\mathcal{E}(K)$ it is easy to present a linear extension operator. But it is not bounded. Following [100], we say that a compact set K has the *extension property* (written EP) if the sequence (4.7) splits, so there is a continuous linear extension operator $W : \mathcal{E}(K) \longrightarrow C^\infty(B)$.

The question of existence of a smooth extension when only values f are given constitutes the Whitney Extension Problem posed in [117]. Given an arbitrary set $E \subset \mathbb{R}^N, p \in \mathbb{N}$, and a function $f : E \longrightarrow \mathbb{R}$, does there exist a function $F \in C^p(\mathbb{R}^N)$ such that $F|_E = f$? In the one-dimensional case, the problem was solved in [117], where Whitney characterized such functions in terms of divided differences of $p-$th order, see Section 11 for the restriction spaces. The number I at the end of title [117] suggests author's desire to continue the research. However, the problem occurs much more difficult for $N \ge 2$. More than in 50 years, Shvartsman in [96] proposed a solution for the case $C^{1,\alpha}(\mathbb{R}^N), \alpha$ is a modulus of continuity, using the *finiteness principle.* To guarantee that f extends to $F \in C^p(\mathbb{R}^N)$ it suffices to find uniformly bounded extensions of f for all $k^\sharp$-element subsets of E, where the finiteness number $k^\sharp$ depends only on N and p. In a series of papers, starting with [28], Fefferman and his followers presented a solution for the general case and considered some related issues such as the computational aspects of the Whitney problem.

5 Extension Operators

Here we review approaches to construct a continuous linear extension operators for differentiable functions. The first such method (weighted sum of reflections) goes back to [73]. Suppose $f \in C^1(\bar{\Omega})$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. Part of $\partial(\Omega)$ belongs to a plane, and the entire region is located on one side of this plane. Assume without loss of generality that $\Omega \subset \{(x,y,z) : z < 0\}$. Then f can be extended by a suitable reflection with respect to the plane $z = 0$. Thus essentially it can be considered as an extension of one-dimensional functions by the operator $E_1 : C^1[0,1] \longrightarrow C^1[-1,1] : f \mapsto F$ with $F(x) = a_1\, f(b_1\, x) + a_2\, f(b_2\, x)$ for $x < 0$, where

$b_1, b_2 \in (-1,0)$ can be selected and a_1, a_2 are solutions of the system $a_1 + a_2 = 1, a_1 b_1 + a_2 b_2 = 1$ that provides differentiability of F at the origin.

Despite the emergence of powerful Whitney operator (4.2), the reflection method was used in [61], where for each finite p a similar operator $E_p : C^p[0,1] \longrightarrow C^p[-1,1] : f \mapsto F$ was presented and in [95] for $p = \infty$. Here, to simplify the corresponding infinite system $\sum_{k=1}^{\infty} a_k b_k^p = 1$, $\sum_{k=1}^{\infty} |a_k| |b_k|^p < \infty, p \in \mathbb{Z}_+$, the values $b_k = -2^k$ were fixed, so the desired operator is considered in a half space. Let $\psi \in C^\infty(\mathbb{R})$ be such that $\psi(x) = 1$ for $x \le 1$, $\psi(x) = 0$ for $x \ge 2$. Then

$$E : C^\infty(\mathbb{R}_+) \longrightarrow C^\infty(\mathbb{R}) : f \mapsto F \;\text{ with }\; F(x) = \sum_{k=1}^{\infty} a_k f(b_k x)\psi(b_k x) \;\text{ for }\; x < 0.$$

An integral version of the reflection method was applied in [98], T.6.5, where an extension operator was presented for a domain $\Omega \subset \mathbb{R}^N$ with the Lipschitz boundary. The operator was defined for the Sobolev space of $C^\infty-$functions that coincides with $C^\infty(\bar{\Omega})$ for a such domain. "The generalized Lichtenstein extension" was considered in [1], Section 11, for similar domains and in [85] for closed set with smooth boundary. Obviously, the reflection method cannot be applied for arbitrary compact sets.

Let us consider three approaches to construct $W : \mathcal{E}(K) \longrightarrow C^\infty(B)$ that can be used for more wide families of sets K. The first method goes back to [79], where for $K = [-1,1]$ Mityagin found in Lemma 25 a basis $(e_n)_{n=0}^{\infty}$ for the space $\mathcal{E}(K)$, extended e_n individually to $\tilde{e}_n \in C^\infty(B)$ and defined in Theorem14 the extension operator as follows

$$W : f = \sum_{n=0}^{\infty} \xi_n \cdot e_n \mapsto \sum_{n=0}^{\infty} \xi_n \cdot \tilde{e}_n. \tag{5.1}$$

This allowed him to construct a basis in the space $C^\infty(\mathbb{R})$ and to show in Theorem 15 the isomorphism $C^\infty(\mathbb{R}) \simeq s^{\mathbb{N}}$. As was shown later in [68], each nuclear space is isomorphic to s^Λ for a suitable index set Λ. In particular, $F-$space $\mathcal{X}$ is nuclear if and only if $\mathcal{X}$ is isomorphic to a subspace of $s^{\mathbb{N}}$.

We see that (5.1) applies to any $\mathcal{E}(K)$ with a basis. Moreover, by T.2.4 in [112], if K has non-empty interior and $\mathcal{E}(K)$ has a dominating norm then $\mathcal{E}(K) \simeq s$. Thus the norms of e_n can be controlled: for sufficiently large p there are $q, Q \in \mathbb{N}$ and positive C_1 and C_2, all depending on p so that $C_1 n^q \le ||e_n||_p \le C_2 n^Q$ for all n.

Furthermore, the approach can be applied to any compact set K with EP, since then the sequence (4.7) splits, so $\mathcal{E}(K)$ is a complemented subspace of s. Following [25], we see that $\mathcal{E}(K)$ has a basis. Thus, the extension operator can always be represented in the form (5.1).

The second method is due to Pawłucki and Pleśniak. In [87], they presented an extension operator in the form

$$W(f)(x) = L_1(f, x; \mathcal{F}_1) \cdot u_{\delta_1}(x) + \sum_{n=1}^{\infty} [L_{n+1}(f, x; \mathcal{F}_{n+1}) - L_n(f, x; \mathcal{F}_n)] \cdot u_{\delta_n}(x). \tag{5.2}$$

Here $\mathcal{F} = (x_{k,n})_{k=1,n=1}^{n,\infty}$ is a Fekete array (the points $\mathcal{F}_n = (x_{k,n})_{k=1}^{n}$ maximize the determinant of corresponding Vandermonde matrix), $L_n(f, \cdot; \mathcal{F}_n)$ is the Lagrange interpolating polynomial with nodes at points from $\mathcal{F}_n$ for $f \in \mathcal{E}(K)$, and for a given $\delta > 0$ the function $u_\delta \in C^\infty(\mathbb{R}^N)$ satisfies $u_\delta = 1$ on K and $u_\delta = 0$ outside of δ neighborhood of K.

The operator initially was considered for uniformly polynomially cuspidal compact sets. Later Pleśniak extended the result to any set with the Markov property (MP). Let us recall this result.

We write $\mathcal{P}_n$ for the set of polynomials $Q : \mathbb{K}^N \longrightarrow \mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of degree at most n. We say that a set K is *Markov* if there is a constant r so that for each $p \in \mathbb{N}$ and $Q \in \mathcal{P}_n$,

$$|\, Q\,|_{p,K} \le C\, n^{rp}\, |\, Q\,|_{0,K}, \tag{5.3}$$

where C depends only on K and p.

Also for any infinite compact set $K \subset \mathbb{K}^N$ we consider the sequence of *Markov's factors*

$$M_n(K) = \inf\{C : |\,\nabla Q\,|_{0,K} \le C\,|\,Q\,|_{0,K} \quad \text{for each } \; Q \in \mathcal{P}_n\}, n \in \mathbb{N}.$$

We see that a set K is Markov if and only if the sequence $(M_n(K))_{n=1}^\infty$ is of polynomial growth. Uniformly polynomially cuspidal compact sets are Markov.

In [88], Pleśniak, following [127], considered the so-called Jackson topology τ_J on $\mathcal{E}(K)$. It is defined by the semi-norms $d_{-1}(f) = |f|_0$, $d_0(f) = E_0(f, K)$, $d_p(f) = \sup_{n\ge1} n^p\, E_n(f, K)$ for $p \ge 1$. Here, $E_n(f, K)$ denotes the best approximation to f on K by $P \in \mathcal{P}_n$ in the uniform norm. By Jackson's theorem, the topology τ_J is not stronger than Whitney topology τ_W. If K is C^∞ *determining* ($f \in C^\infty(\mathbb{R}^N)$ and $f|_K = 0$ imply $D^\alpha f|_K = 0$ for all $\alpha \in \mathbb{Z}_+^N$), then τ_J is Hausdorff and $(\mathcal{E}(K), \tau_J)$ has the dominating norm property ([3], p. 36).

Let K be C^∞ determining. Then, according to T.3.3 in [88], the following are equivalent

(a) The operator $W : (\mathcal{E}(K), \tau_J) \longrightarrow C^\infty(B)$ given by (5.2) is continuous.

(b) The topologies τ_J and τ_W coincide on $\mathcal{E}(K)$.

(c) The space $(\mathcal{E}(K), \tau_J)$ is complete.

(d) K is Markov.

Thus every Markov set has EP. However, by [36], the inverse implication is not valid. In the case of non-Markov compact sets with EP, the operator (5.2) is not continuous in τ_J. Nevertheless, this does not exclude the possibility of its boundedness in stronger topology τ_W. Recently, in [56] it was shown that the operator (5.2) is continuous in τ_W for the set K^2. This set is the worst (with respect to growth of Markov's factors) in a class of Cantor-type sets with EP, see the next section for more details. The gist of proof is that the rapid growth of $(M_n(K^2))_{n=1}^\infty$ can be compensated by ultra-fast decrease of $E_n(\cdot, K^2)$ as $n \to \infty$. Despite this result, the author does not believe that this argument can be used for every K with EP. One can propose to consider a mixed K, with a dense part and a sparse part, for example $K = [-1, 0] \cup K^2$ and compare the behavior of $(M_n(K))_{n=1}^\infty$ and $(E_n(\cdot, K))_{n=1}^\infty$. Perhaps in the general case one could use a local version of this operator, as in [3].

The third method that caught our attention was proposed in [30], where Frerick, Jordá, and Wengenroth showed that the classical Whitney extension operator (4.2) for the space of jets of finite order can be generalized to the case $\mathcal{E}(K)$. The authors replaced Taylor's polynomials at a_j in Whitney's construction (4.3) with interpolation using certain local measures $\mu_{\alpha,j}$ supported on K and concentrated near a_j with a controllable relation between j and specially chosen maximums of $|\alpha|$. The corresponding operator

$$W(f)(x) = \sum_j \psi_j(x) \sum_{|\alpha|\le m(j)} \frac{1}{\alpha!} \int f(x) d\mu_{\alpha,j}(x - x_j)^\alpha \tag{5.4}$$

is bounded provided the following local version of the Markov inequality on K: there exist $r \ge 1$ and a sequence $(c_n)_{n=0}^\infty$ with $c_n \ge 1$ such that for each $Q \in \mathcal{P}_n$, $0 < \varepsilon < 1$ and $x_0 \in K$

$$|\nabla Q(x_0)| \le c_n\, \varepsilon^{-r} \sup\{|Q(x)| : x \in K, |x - x_0| \le \varepsilon\}.$$

On the other hand, the above operator satisfies the following *linear tame* continuity estimates: for each p and ε there is a constant C such that

$$|W(f)|_p \le C\,||f||_{(r+\varepsilon)p} \quad \text{for} \;\; f \in \mathcal{E}(K).$$

Here r is the exponent of the above local Markov inequality. It is interesting to check, whether is it possible to sacrifice this strong type of continuity for W in order to extend the method to any K with EP?

Another question that interests the author: are these methods of linear extension different? Is there a unifying approach that brings them together? At least for K^2 and for $K = [-1,1]$, the operator (5.1) and (5.2) coincide, see Conclusion 4 and Example 7.2 in [56].

6 Characterizations of the Extension Property

As mentioned in Section 2, there is an important linear topological characterization of EP: *a compact set K has the extension property if and only if the space $\mathcal{E}(K)$ has a dominating norm (satisfies (2.1)).*

Here we discuss the geometric conditions for EP. Let us start from the simplest example of K without EP, a singleton, see [79], Proposition 21, in [14] it is attributed to Grothendieck. If $K = \{0\}$ then the sequence (4.7) has the form

$$0 \longrightarrow \mathcal{F}(\{0\},[-1,1]) \xrightarrow{\;i\;} C^\infty[-1,1] \xrightarrow{\;\pi\;} \mathcal{E}(\{0\}) \longrightarrow 0.$$

If there exists W, a right inverse of π then there are q and C with $|W(f)|_{0,[-1,1]} \le C\,||f||_q$ for each $f \in \mathcal{E}(\{0\})$. For the jet f corresponding to x^{q+1} we have $||f||_q = 0$ but the function $W(f) \in C^\infty[-1,1]$ cannot be zero, a contradiction.

Similarly, the space ω cannot be a complemented subspace of $C^\infty[-1,1]$, which has a continuous norm, whereas any continuous seminorm on ω can not be a norm.

The same conclusion about the absence of EP can be drawn for every compact set K with an isolated point. For such sets, the space $\mathcal{E}(K)$ obviously has no dominating norm.

On the other hand, by means of (5.1), Mityagin proved that $K = [-1,1] \subset \mathbb{R}$ has EP. In [79], p.124, he posed the following problem (in our terms): *What is a geometric characterization of the extension property?*

At the beginning, the main object for generalizations was $\bar{\Omega}$, where Ω is a domain in $\mathbb{R}^N$. Without loss of generality, we consider only bounded domains, so $K = \bar{\Omega}$ is compact. If $\partial\Omega$ is sufficiently regular then K has EP. This has been shown by many people in different ways.

Let us write $\Omega \in Lip\,\alpha$ for $0 < \alpha \le 1$ if the boundary of Ω is locally the graph of a function of Lipschitz class α. For the domains considered below, $C^\infty(\bar{\Omega}) \simeq \mathcal{E}(\bar{\Omega})$.

In [85], for domains with $C^\infty-$smooth boundary and in [98] for $Lip\,1$ boundary some modifications of the reflection method were used. Bierstone in [13] considered the case $\Omega \in Lip\,1$ in the framework of the general lifting problem. In [106] and [10] it was shown that $C^\infty(\bar{\Omega}) \simeq s$ for a domain with $C^\infty-$smooth boundary by constructing a basis from eigenvectors of an elliptic differential operator. This isomorphism was proved using bases from orthogonal in $L_2(\Omega)$ polynomials for $\Omega \in Lip\,1$ in [127] and $\Omega \in Lip\,\alpha$ in [33], T.1.1. Tidten showed in [100] that $\mathcal{E}(\bar{\Omega})$ has the (DN) property for $\Omega \in Lip\,\alpha$ but in the case $\psi_0(x) = \exp(-1/x)$ the space $\mathcal{E}(\bar{\Omega}_{\psi_0})$ does not have EP. Before, in [127], it was shown that the sequence of polynomials $Q_{n+1} = y\,(1-x)^n$ violates (5.3) on $\bar{\Omega}_{\psi_0}$. Also, it was known for a compact set $K \subset \mathbb{C}$, that the Hölder continuity of the Green function $g_{\mathbb{C}\setminus K}$ implies that K is Markov. These results were combined in [33], T.1.3: suppose $\psi \in C([0,1])$ is an increasing convex function with $\psi(0) = 0$ and the domain Ω_ψ is defined as in (3.3). We do not expect another cusp in Ω_ψ, so let at the point $(1,\psi(1))$ the boundary of Ω_ψ is Lipschitz. Then the following are equivalent:

(a) $\Omega_\psi \in Lip\,\alpha$ for some $0 < \alpha \le 1$.

(b) The compact set $\bar{\Omega}_\psi$ is Markov.

(c) The spaces $C^\infty(\bar{\Omega}_\psi)$ and s are isomorphic.

(d) The Green function $g_{\mathbb{C}\setminus K}$ is Hölder continuous.

It was natural to assume that the Markov property is a (though not geometric) characteristics of EP. However, in [36] it was shown that in general MP is not necessary for EP. The following set is a sequence of closed intervals tending to a point. The identification of intervals with Donald Duck paw prints justifies the name *running duck set*.

Example 6.1. *Let* $b_n = \exp(-M^n)$ *for* $M \ge 3, a_n = b_n - b_{n+1}$ *and* $K = \{0\} \bigcup \bigcup_{n=0}^\infty [a_n, b_n]$. *Then* $\mathcal{E}(K)$ *has the DN property, but* K *is not Markov, see* [36] *for more details. It should be noted that this compact set also does not satisfy the version of the local Markov inequality introduced in* [15] *to characterize the existence of an extension operator "with homogeneous loss of differentiability". It is a kind of the linear tame continuity, see* 1.10 *and* 1.36 *in* [15] *for the corresponding definitions.*

The analysis of EP for sets of the above type was done in [37]. Suppose $0 < \cdots < b_{n+1} < a_n < b_n < \cdots < b_1 < 1$. Let $I_n = [a_n, b_n]$ and $K = \{0\} \bigcup \bigcup_{n=1}^\infty I_n$. By ℓ_n we denote the length of $I_n, \hbar_n = a_n - b_{n+1}$ is the distance between I_n and I_{n+1}. We assume that $\ell_n \searrow 0$, $\hbar_n \searrow 0$, $\ell_n \le \hbar_n$ and there exists Q so that $\hbar_n \ge b_n^Q$ for all n. By Theorem 3 in [37], the following three statements are equivalent:

(a) There are M and n_0 such that $\ell_n \ge \hbar_{n-1}^M$ for $n \ge n_0$.

(b) K has EP.

(c) There exists $\alpha \ge 1$ so that K is $\alpha-$perfect. (The definition is given below).

More general sufficient condition for EP of $K \subset \mathbb{R}^N$ in geometric terms was proposed by Frerick in [29], p. 124: there exist $\varepsilon_0, \rho > 0, r \ge 1$ so that for each point y in the boundary of K and $0 < \varepsilon < \varepsilon_0$ there is x satisfying two conditions: $|x - y| < \varepsilon$ and the ball with center at point x and radius $\rho\varepsilon^r$ belongs entirely to K.

All considered above compact sets with the extension property have non-empty interior. What is known about K with EP and $IntK = \emptyset$? Suppose $K \subset \mathbb{R}$. Clearly, K with EP must be perfect. Tidten proved that uniformly perfect compact sets have EP. Following [101], we say that a compact set $K \subset \mathbb{R}$ is a perfect set of class α for $\alpha \ge 1$ (written $K \in (\alpha)$) if there are constants $C \ge 1$ and $\delta_0 > 0$ such that for any point y in K there is a sequence $(x_n) \subset K$ with $|y - x_1| \ge \delta_0, |y - x_n| \searrow 0$ and $|y - x_n|^\alpha \le C\,|y - x_{n+1}|$ for all n. Uniformly perfect compact sets are exactly sets of the class (1). By [101],

$$K \in (1) \Rightarrow K \quad \text{has} \quad EP \Rightarrow K \in (\alpha) \quad \text{for some} \quad \alpha \ge 1.$$

Since the classical Cantor ternary set K_0 is uniformly perfect, it became the first example of the sets discussed here. Later, in [12] it was shown that $g_{\mathbb{C}\setminus K_0}$ is Hölder continuous, so K_0 is Markov and has EP, another proof.

In development of [101], in [38] the extension problem for the family K^α of Cantor-type sets was considered. First, we define geometrically symmetric sets of this kind. Let $\mathcal{L} = (\ell_s)_{s=0}^\infty$ be a sequence such that $\ell_0 = 1$ and $0 < 3\ell_{s+1} \le \ell_s$ for $s \in \mathbb{Z}_+$. For $E_0 = I_{0,1} = [0,1]$ and given E_s, a union of 2^s closed *basic* intervals $I_{j,s}$ of length ℓ_s, we obtain E_{s+1} by replacing each $I_{j,s}$ for $1 \le j \le 2^s$ by two adjacent sub-intervals $I_{2j-1,s+1}$ and $I_{2j,s+1}$. Then $\hbar_s = \ell_s - 2\,\ell_{s+1}$ is the distance between them. By assumption, $\hbar_s \ge \ell_{s+1}$ for each s. A Cantor-type set associated with the sequence $\mathcal{L}$ is $K(\mathcal{L}) = \bigcap_{s=0}^{\infty} E_s$.

Now fix $\alpha > 1$ and choose ℓ_1 with $2\,\ell_1^{\alpha-1} < 1$. By K^α we denote the Cantor-type set associated with $(\ell_s)_{s=0}^{\infty}$ for $\ell_{s+1} = \ell_s^{\alpha} = \cdots = \ell_1^{\alpha^s}$ for $s \in \mathbb{N}$. It's easy to see that $K^\alpha \in (\alpha) \setminus (\beta)$ for $\beta < \alpha$. All K^α are not Markov (Prop.1 in [89]). In [38] it was proved that K^α has EP for $\alpha < 2$ and does not have it for $\alpha > 2$. The value $\alpha = 2$ is critical for such sets also in potential theory. Let $CapK$ be the logarithmic capacity of a compact set $K, \mu_{h_0}(K)$ be its logarithmic measure. Below we will discuss this and other Hausdorff measures regarding EP. By Corollary 1 in [38], for $\alpha \ne 2$, the following are equivalent:

(a) $\alpha < 2$.

(b) $CapK^\alpha > 0$.

(c) $\mu_{h_0}(K^\alpha)$ is positive (in fact infinite).

(d) $\mathbb{C} \setminus K^\alpha$ is regular with respect to the Dirichlet problem.

(e) K^α has the extension property.

Of course, such global property as polarity cannot be equivalent to the lack of EP, which is local in its nature: a compact set K has EP if it is not small (in some unknown for us sense) in any neighborhood of each $x \in K$. The same reason is valid for Hausdorff measures, so below we will consider densities of such measures. Moreover, the above condition (b) does not characterize EP even for Cantor-type sets. By [41], the polar set K^2 has EP.

Condition (d) above means that the Green function $g_{\mathbb{C}\setminus K^\alpha}$ is continuous throughout $\mathbb{C}$. The following example from [38] shows that regularity cannot be a general characteristic of EP either.

Example 6.2. *Consider two sets* $K_j = \{0\} \bigcup \bigcup_{n=2}^{\infty} [1/n, b_n^{(j)}]$ *with* $b_n^{(1)} = 1/n + n^{-n}$ *and* $b_n^{(2)} = 1/n + n^{-n^2}$. *The set* K_2 *is thin at the origin, whereas* K_1 *is regular. This can be checked using Wiener's criterion and estimation of capacities for union of sets, see, e.g.* [90], *T.* 5.4.1 *and T.* 5.1.4. *By* [37], *T.* 3, *both compact sets do not have the extension property.*

The result from [38] was generalized in [46] to the class of sets $K^{(\alpha)}$. Now α is a sequence $(\alpha_s)_{s=2}^{\infty}$, with $\inf_s \alpha_s > 1$. Fix $0 < \ell_1 < 1$. Let $K^{(\alpha)}$ be the Cantor set associated with the sequence

$$\ell_0 = 1, \ell_1, \ell_2 = \ell_1^{\alpha_2}, \cdots, \ell_s = \ell_1^{\alpha_2\alpha_3\cdots\alpha_s}, \cdots. \tag{6.1}$$

Of course, K^α is $K^{(\alpha)}$ for the constant sequence. Let $\pi_{n,0} := 1$ and $\pi_{n,k} = 2^{-k}\,\alpha_{n+1}\alpha_{n+2}\cdots\alpha_{n+k}$ for $n, k \in \mathbb{N}$. Then the condition

$$\sum_{k=0}^{s+1} \pi_{n,k} \Big/ \sum_{k=0}^{s} \pi_{n,k} \rightrightarrows 1 \quad \text{as} \quad s \to \infty \tag{6.2}$$

(the symbol $\rightrightarrows$ denotes the convergence that is uniform with respect to n) is equivalent to the following requirement in geometric terms

$$\forall M > 0 \quad \exists s_M: \quad \ell_{n+s}^M > \ell_n^{2^s}\,\ell_{n+1}^{2^{s-1}} \cdots \ell_{n+s}, \quad \forall n \ \ \forall s \ge s_M. \tag{6.3}$$

By [46], $K^{(\alpha)}$ has the extension property if and only if the condition (6.2) is fulfilled.

The set $K^{(\alpha)}$ is polar if and only if $\sum_{k=0}^{\infty} \pi_{1,k} = \infty$. The sum $\sum_{k=0}^{s} \pi_{n,k}$ is related to the Robin constant (minimal logarithmic energy) of the set $K^{(\alpha)} \cap I_{j,n}$ for each fixed $1 \le j \le 2^n$.

Thus, from the point of view of potential theory, the condition (6.2) means a regular (and uniform in n) growth of the partial sums corresponding to the Robin constants of the sets indicated above, up to their limit values (possibly infinite). Since the calculation of minimum energies for such sets is quite difficult, the author introduced in [50] a family of sets $K(\gamma)$, for which many values related to potential theory can be found exactly.

Let $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < \frac{1}{4}$, $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Take $P_2(x) = x(x-1)$ and, recursively, $P_{2^{s+1}} = P_{2^s} \cdot (P_{2^s} + r_s)$. Then P_{2^s} has 2^{s-1} minima with equal values $-r_{s-1}^2/4$ so the line $y = -r_s$ intersects the graph of P_{2^s} at 2^s points, which together with the zeros of P_{2^s} form the zeros of $P_{2^{s+1}}$. The set $E_s = \{x \in \mathbb{R} : |P_{2^s}(x) + r_s/2| \le r_s/2\}$ is a union of 2^s closed disjoint intervals $I_{j,s}$ with $\max_j |I_{j,s}| \to 0$ as $s \to \infty$. Clearly, $E_{s+1} \subset E_s$. We define $K(\gamma)$ as $\bigcap_{s=0}^{\infty} E_s$. Thus, $K(\gamma)$ is the intersection of inverse polynomial images $(\frac{2}{r_s} P_{2^s} + 1)^{-1}([-1,1])$, which gives the corresponding Chebyshev polynomials $T_{2^s, K(\gamma)} = P_{2^s} + r_s/2$.

On the other hand, $K(\gamma) = \bigcap_{s=0}^{\infty} \overline{D}_s$ with the decreasing sequence of complex level domains $D_s = \{z \in \mathbb{C} : |P_{2^s}(z) + r_s/2| < r_s/2\}$. The Green functions $g_{\mathbb{C}\setminus\overline{D}_s}$ are known, so by the Harnack Principle, $g_{\mathbb{C}\setminus K(\gamma)}(z) = \lim_{s\to\infty} 2^{-s} \log |P_{2^s}(z)/r_s|$. This allows us to show that $Cap(K(\gamma)) = \exp(\sum_{s=1}^{\infty} 2^{-s} \log \gamma_s)$. Recall that the exact value of the classical Cantor ternary set is not known.

The most interesting feature of $K(\gamma)$ is that, by [51], T.8.3, $\mu_{K(\gamma)} = \mu_{h_{K(\gamma)}}$ provided $\gamma_s < 1/32$ and $\sum_{s=1}^{\infty} \gamma_s < \infty$. That is the equilibrium measure and the corresponding Hausdorff measure coincide on the set. Usually these two natural measures on Cantor-type sets are mutually singular. The value $\delta_s = \gamma_1 \cdots \gamma_s$ is responsible for lengths $\ell_{j,s}$ of basic intervals in the Cantor procedure, see (3) in [51], whereas $B_k = 2^{-k-1} \cdot \log \frac{1}{\delta_k}$ gives the Robin constant of the set as $Rob(K(\gamma)) = \sum_{k=1}^{\infty} B_k$. By Theorem 5.3 in [51], provided the above conditions, the set $K(\gamma)$ has EP if and only if

$$\frac{\sum_{k=n}^{n+s} B_k}{\sum_{k=n}^{n+s-1} B_k} \rightrightarrows 1 \quad \text{as} \quad s \to \infty \quad \text{uniformly with respect to} \quad n. \tag{6.4}$$

The condition includes polar sets, for example the values $\gamma_k = \exp(-2^k M)$ for sufficiently large M give a polar $K(\gamma)$ with EP.

The above result shows that there is no complete description of EP in terms of densities for Hausdorff contents or related characteristics. Let us recall the meaning of these terms.

Let $h : (0,T) \to (0,\infty)$ be a dimension function, that is continuous, non decreasing function with $h(t) \to 0$ as $t \to 0$. The $h-$Hausdorff content of $E \subset \mathbb{R}$ is defined as

$$M_h(E) = \inf\{\sum h(|G_i|) : E \subset \bigcup G_i\}$$

and the $h-$Hausdorff measure of E is

$$\mu_h(E) = \liminf_{\delta\to 0}\{\sum h(|G_i|) : E \subset \bigcup G_i, |G_i| \le \delta\}.$$

Here we consider coverings of E by open intervals. An important *logarithmic* measure is defined by

$$h_0(t) = \left(\log \frac{1}{t}\right)^{-1} \quad \text{for} \quad 0 < t < 1, \tag{6.5}$$

which is the inverse to ψ_0 from Example 3.2.

We write $h_1 \prec h_2$ if $h_1(t) = o(h_2(t))$ as $t \to 0$. Let $h_1 \approx h_2$ if $C^{-1} h_1(t) \le h_2 \le C h_1(t)$ for some constant $C \ge 1$ and $0 < t \le t_0 < T$.

A set E is called *dimensional* if there is h so that $0 < \mu_h(E) < \infty$. Then we say that E is an $h-$set. The set $K(\gamma)$ is dimensional. Its dimension function $h_{K(\gamma)}$ can be constructed as in [84], and (in the non-polar case) it can be chosen such that the restriction of μ_h to $K(\gamma)$ coincides with the equilibrium measure of the set, as mentioned above.

By Proposition 8.4 in [51], there are dimension functions $h_2 \prec h_1$ and two $K(\gamma)$ sets K_1, K_2, where K_j is an h_j-set for $j \in \{1,2\}$, such that the smaller set K_1 has EP, whereas the larger set K_2 does not.

The above characterization of EP for K^α suggests that one should consider dimension functions close to h_0. For this reason, dimension functions of the form $h = h_0^\rho$ for different monotone functions ρ with mild regularity restrictions were discussed in [51]. But for such h, by Proposition 9.1 in [51], M_h does not distinguish EP for running duck sets.

Global characteristics as Hausdorff measures or Hausdorff contents cannot be applied in general for characterization of EP. Let us consider the lower density of M_h. Given a dimension function h and a compact set K, let $\phi_h(K) := \liminf_{r\to 0} \inf_{x\in K} \frac{M_h(K\bigcap B(x,r))}{M_h(B(x,r))}$, where $B(x,r) = [x-r, x+r]$. A similar version of density for μ_h is trivial since then the denominator is infinite for $t \prec h$.

For K from a wide family of running duck sets, one can present h so that K has EP if and only if $\phi_h(K) > 0$, [51], Pr.9.3. But for any dimension function of the type under consideration, there is such a set K with EP and a set $K(\gamma)$ without EP despite $\phi_h(K) < \phi_h(K(\gamma))$.

Also, the extension property cannot be characterized in terms of the growth of Markov factors for the set. By Proposition 10.1 in [51], there are two $K(\gamma)$ sets K_1 with EP and K_2 without it, such that $M_n(K_1)$ grows essentially faster than $M_n(K_2)$ as $n \to \infty$.

We observe some similarity of the above results with the state in potential theory. Let K be a compact subset of $\mathbb{C}$. It is well known that $\mu_{h_0}(K) = 0$ implies polarity of K. On the other hand, by Nevanlinna [84], $\mu_h(K) > 0$ implies that $Cap(K) > 0$ for each h with $\int_0 \frac{h}{r}dr < \infty$. However, by [109], between h with $\int_0 \frac{h}{r}dr < \infty$ and h_0 there is a zone of uncertainty. Namely, in this zone, there are two dimension functions h_1, h_2 with $h_2 \prec h_1$, non-polar K_1 and polar K_2, such that $\mu_{h_2}(K_2) > 0$, whereas $\mu_{h_1}(K_1) < \infty$, so $\mu_{h_2}(K_1) = 0$. This means that there is no general characterization of polarity of sets in terms of Hausdorff measures.

Similarly, if we restrict ourselves with dimensional Cantor-type sets, then the scale for growth rate of dimension functions h can be decomposed into three zones with respect to the extension property. For h from the first zone of small growth, if $0 < \mu_h(K)$ then the set K has EP. For h from the zone of fast growth, $\mu_h(K) < \infty$ implies the absence of EP for K. And there is a zone of uncertainty $\mathcal{Z}_h$ between them.

And for Markov factors (now for all compact sets), the scale of their growth rate can be decomposed into three zones with respect to the extension property. For $(M_n)_{n=1}^\infty$ from the first zone of slow growth, if $M_n(K) \prec M_n$ then K has EP. For $(M_n)_{n=1}^\infty$ from the zone of fast growth, if $M_n \prec M_n(K)$ then K does not have EP. And there is a zone of uncertainty $\mathcal{Z}_M$.

Of course, it is interesting to find in both cases the boundaries of these zones. At least for $\mathcal{Z}_M$, its lower bound is not less than the polynomial growth. One can conjecture that its upper bound is given by $(M_n(K^2))_{n=1}^\infty$. By Theorem 4.5 in [56], this is exponential growth $\exp(2n)$.

Despite the disappointing examples from [51], the author hopes that the generalization of Lemma 2 in [37], (a kind of $\alpha-$perfectness provided by groups of points) can serve as a geometric characterization of EP. The following may be offered, at least as a sufficient condition of EP for $K \subset \mathbb{R}: \exists \varepsilon_0, m:$ $\forall x \in K, \forall 0 < \varepsilon \le \varepsilon_0, \forall q\ \exists r > q$ so that there are points $(x_k)_{k=1}^r \subset K \bigcap [x_0-\varepsilon, x_0+\varepsilon]$ with $|x_k - x_0| \nearrow$, where $(x_k)_{k=1}^q$ are "weakly uniformly distributed" in $[x_0 - \varepsilon^m, x_0 + \varepsilon^m]$ whereas $(x_k)_{k=q+1}^r$ are "weakly uniformly distributed" in $[x_0-\varepsilon, x_0+\varepsilon]$. Here, "weakly uniformly distributed" means that $|x_1 - x_0|$ and max $|x_k - x_0|$ are comparable using the same constant depending only on K. Applying the argument of Lemma one can obtain the additive form of the (DN) property.

7 Bases in $C^\infty[-1,1]$

There is a variety of bases in the space $C^\infty[-1,1]$. The first idea was to use orthogonal in $L_{2,\mu}$ polynomials with a proper measure μ and show that the corresponding orthogonal expansion for a given f converges also in more strong topology of the space. The classical work here is [79], L.25, where Mityagin proved that the Chebyshev polynomials of the first kind $(T_n)_{n=0}^\infty$ form a topological basis in the space $C^\infty[-1,1]$. Since below we shall consider the scaling version of $(T_n)_{n=0}^\infty$, let us recall basic facts about them.

Let $T_n(x) = \cos(nt)$ for $x = \cos t$, where $0 \le t \le \pi, n \in \mathbb{Z}_+$. The polynomials are orthogonal with respect to the equilibrium measure $\mu_{[-1,1]}$ with $d\mu = \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}$ for $|x| < 1$. Hence the functionals $\xi_n(f) = c_n \int_0^\pi f(\cos t)\cos nt dt$ with $c_0 = \frac{1}{\pi}$ and $c_n = \frac{2}{\pi}$ for $n \in \mathbb{N}$, are biorthogonal to $(T_n)_{n=0}^\infty$. For each $f \in C^\infty[-1,1]$, the sequence $(\xi_n(f))_{n=0}^\infty$ rapidly decreases, while, by the classical Markov inequality, $|T_n|_{p,[-1,1]} \le n^{2p}$. It follows that the series $\sum_{n=0}^\infty \xi_n(f)\cdot T_n$ converges to f in the desired topology.

Later, in a similar way, it was shown in [59] that the Legendre polynomials and in [8] that any Jacobi polynomials have the basis property for the space, so the measures with $d\mu = (1+x)^\alpha(1-x)^\beta$ with $\alpha, \beta > -1$ were considered.

This method can be generalized to a wide class of measures on $[-1,1]$. Let μ be a finite Borel measure with $supp(\mu) = [-1,1]$. By $(P_{n,\mu})_{n=0}^\infty$ we denote the polynomials obtained by the orthonormalization of monomials in $L_{2,\mu}$. Assume that the their uniform norms have at most polynomial growth, so there are constants C, r with

$$|P_{n,\mu}|_{0,[-1,1]} \le C\, n^r \quad \text{for} \quad n \in \mathbb{N}. \tag{7.1}$$

This is a special case of the so-called Bernstein-Markov measures on $[-1,1]$, for which on the right-hand side we have a term of the sub-exponential growth sequence.

Theorem 7.1. *Let μ be as above. Then $(P_{n,\mu})_{n=0}^\infty$ is a basis in the space $C^\infty[-1,1]$.*

Proof. Fix $p \in \mathbb{N}$. Then, as above, $|P_{n,\mu}|_{p,[-1,1]} \le Cn^{r+2p}$.

The functionals $\xi_n(f) = \int f(x)P_{n,\mu}(x)d\mu(x), n \in \mathbb{Z}_+$, are biorthogonal to $(P_{n,\mu})_{n=0}^\infty$. Suppose $f \in C^\infty[-1,1]$. Given r from (7.1), let $q > r, n > q$ and Q_{n-1} be the $n-1$–st degree polynomial of best approximation to f. By Jackson theorem (see, e.g 5.1.5 in [103]), $E_{n-1}(f,[-1,1]) \le C_q/n^q$, where C_q is a constant depending only on q. Then by ortogonality and (7.1), $|\xi_n(f)| \le \int |f(x) - Q_{n-1}(x)|\,|P_{n,\mu}(x)|\,d\mu(x) \le C\,C_q\,\mu([-1,1])\,n^{r-q}$.

Thus $(\xi_n(f))_{n=0}^\infty \in s$ and the value $q \ge 2p+2r+2$ ensures the convergence of the series $\sum_{n=0}^\infty \xi_n(f)\cdot P_{n,\mu}$ in $C^\infty[-1,1]$. By the condition on support of μ, the functionals $(\xi_n(f))_{n=0}^\infty$ are total over the space, therefore the series converges exactly to f. □

Of course, $C^\infty[-1,1]$ also has a wavelet basis. Kilgor and Prestin suggested in [65] a system of wavelets constructed from the Chebysev polynomials $(T_n)_{n=0}^\infty$. Its basis property was shown in [47].

The following approach will be important in constructing bases of the Whitney space $\mathcal{E}(K)$ for a sparse compact set K. It can also be used for $K = [-1,1]$.

For a compact set $K \subset \mathbb{R}$ and a sequence of distinct points $(x_n)_1^\infty \subset K$, let $e_0 \equiv 1, e_n(x) = \prod_1^n(x - x_k)$ for $n \in \mathbb{N}$. Let $\mathcal{X}(K)$ be a Fréchet space of continuous functions on K, containing all polynomials. Now ξ_n for $n \in \mathbb{Z}_+$ is a linear functional on $\mathcal{X}(K)$ given as the divided difference $\xi_n(f) = [x_1, x_2, \cdots, x_{n+1}]f$. For the definition and properties of the divided differences, see, e.g. [20], 4.7. The biorthogonality of $(e_n, \xi_n)_{n=0}^{\infty}$ is obvious. In addition, if K is perfect and the points of $(x_n)_1^{\infty}$ are dense in K then the functionals $(\xi_n)_{n=0}^{\infty}$ are total over $\mathcal{X}(K)$.

If for each $f \in \mathcal{X}(K)$ the series $\sum_{k=0}^{\infty} \xi_k(f)\, e_k$ converges to f then we say that $(e_n)_{n=0}^{\infty}$ is an *interpolating* bases for the space. Indeed, the partial sums $L_n(f,x) = \sum_{k=0}^{n} \xi_k(f)\, e_k$ are exactly the interpolating polynomial of f. Here $L_n : \mathcal{X}(K) \to \mathcal{P}_n : f \mapsto L_n(f, \cdot)$ is the projection and its $|\cdot|_{0,K}$ norm is called the Lebesgue constant, written $\Lambda_n(K)$. The characters of growth of $\Lambda_n(K)$ and Markov factors $M_n(K)$ as $n \to \infty$ are crucial for the basis property of the system.

Theorem 7.2. *Suppose that a compact set $K \subset \mathbb{R}$ is Markov and there is a sequence of distinct points $(x_n)_1^{\infty}$ that are dense in K such that $(\Lambda_n(K))_{n=1}^{\infty}$ has a polynomial growth. Then $(e_n)_{n=0}^{\infty}$ is an interpolating basis in the Whitney space $\mathcal{E}(K)$.*

Proof. By (5.3),

$$|\,\xi_n(f)\,| \cdot |\,e_n\,|_{p,K} = |L_n(f) - L_{n-1}(f)|_{p,K} \le C\, n^{rp}\, |L_n(f) - L_{n-1}(f)|_{0,K}.$$

Let Q_n be the $n-$th degree polynomial of best approximation to f on K, so $|f - Q|_{0,K} = E_n(f,K)$. Using the standard argument of Lebesgue's lemma, we have $|L_n(f) - f|_{0,K} \le |L_n(f - Q_n)|_{0,K} + |f - Q_n|_{0,K} \le [\Lambda_n(K) + 1]\, E_n(f,K)$. Therefore

$$|\,\xi_n(f)\,| \cdot |\,e_n\,|_{p,K} \le C\, n^{rp}\, [\Lambda_n(K) + 1][E_n(f,K) + E_{n-1}(f,K)],$$

which is a term of convergent series since $E_n(f,K) \le E_n(F,[-1,1])$ for any extension of f to $F \in C^{\infty}[-1,1]$ and the sequence $(E_n(F,[-1,1]))_{n=1}^{\infty}$ rapidly decreases. □

Since, by [16] the interval $[-1,1]$ contains $(x_n)_1^{\infty}$ with a moderate growth of the corresponding Lebesgue constants, the space $C^{\infty}[-1,1]$ possesses an interpolating basis ([49]).

In view of the isomorphism

$$C^{\infty}[-1,1]^N \simeq \underbrace{C^{\infty}[-1,1]\,\hat{\otimes}_{\pi}\, C^{\infty}[-1,1]\,\hat{\otimes}_{\pi} \cdots \hat{\otimes}_{\pi}\, C^{\infty}[-1,1]}_{N}$$

([58], Ch.2, T.13) all above bases can be used for the multivariate case. Indeed, let the spaces $\mathcal{X}$ and $\mathcal{Y}$ have bases $(e_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$, respectively. It is easily seen that $(e_n \otimes f_k)_{n,k=1}^{\infty}$ is a basis in $\mathcal{X}\hat{\otimes}_{\pi}\mathcal{Y}$.

A polynomial basis $(P_n)_{n=0}^{\infty}$ in a function space is called a Faber (or strict polynomial) basis if $deg\, P_n = n$ for each n. All above bases in $C^{\infty}[-1,1]$ are Faber. For a special non-polynomial basis in $C^{\infty}[0,1]$ that was used in [42] to construct a basis in the space of $C^{\infty}-$functions on a graduated sharp cusp with arbitrary sharpness, see the next section.

The idea of using orthogonal bases in the spaces of $C^{\infty}-$function has been successfully applied to other sets as well. In [106] (see also [10]) a basis from eigenvectors of a certain self-adjoint differential operator can be constructed in $C^{\infty}(\bar{\Omega})$, where Ω is a domain in $\mathbb{R}^N$ with a smooth boundary. Zeriahi in [126] considered a Markov compact set $K \subset \mathbb{R}^N$ and a Borel measure μ on K with the following density property: there are positive constants C, γ and r_0 so that for each $x \in K$ and $0 < r < r_0$ the condition $\mu(K \bigcap B(x,r)) \ge C\, r^{\gamma}$ holds. Here $B(x,r)$ is the ball with center x and radius r. As above, let $(P_{n,\mu})_{n=0}^{\infty}$ be polynomials obtained by the Hilbert-Schmidt procedure in $L_{2,\mu}$ applied to all monomials in $\mathbb{R}^N$ arranged in a sequence such that their degrees are non-decreasing. Then $(P_{n,\mu})_{n=0}^{\infty}$ is a basis in $\mathcal{E}(K)$.

8 Basis in $\mathcal{E}(K)$ for the Sharp Cusp

The first example of NF-space without basis from [128] was simplified in [21], where the space $C^\infty(\bar{\Omega}_{\psi_0})$ for the domain (3.3) has been proposed as a possible candidate for a function NF-space without basis. The analogue of this space for one-dimensional sets is $\mathcal{E}(K)$ for $K = \{0\} \bigcup \bigcup_{k=1}^\infty [a_k, b_k]$, where the lengths of intervals are much smaller than the distances between them. Both spaces contain subspaces $C_0^\infty(\bar{\Omega}_{\psi_0})$ and $\mathcal{E}_0(K)$ respectively, where the index 0 in notation means that the subspace contains only functions that are flat at the origin. It was not difficult to construct bases in these subspaces ([70] and [34]). We present here the basis in $\mathcal{E}_0(K)$.

Let $I_k = [a_k, b_k] = [x_k - \delta_k, x_k + \delta_k]$ and $\hbar_k = a_k - b_{k+1}$. Suppose that $\hbar_k \searrow 0$ and $\delta_k \searrow 0$. We will denote by T_{nk} the restriction to I_k of the scaling Chebyshev polynomial, that is $T_{nk}(x) = T_n(\frac{x-x_k}{\delta_k})$, $x \in I_k$ and $T_{nk} = 0$ for $x \in K \setminus I_k$. In its turn, let

$$\xi_{0k}(f) = \frac{1}{\pi}\int_0^\pi f(x_k + \delta_k \cos t)dt \quad \text{and} \quad \xi_{nk}(f) = \frac{2}{\pi}\int_0^\pi f(x_k + \delta_k \cos t)\cos nt dt$$

for $n \in \mathbb{N}$. The functionals (ξ_{nk}) are, clearly, biorthogonal to (T_{nk}).

Let $X_k = \{f \in \mathcal{E}_0(K) : \operatorname{supp} f \subset I_k\}$. Then the flatness of functions at the origin implies the decomposition $\mathcal{E}_0(K) = (\oplus X_k)_s$, which means that each $f \in \mathcal{E}_0(K)$ has a unique representation $f = \sum_{k=1}^\infty f_k$ with $f_k \in X_k$ so that for every $p \in \mathbb{N}$ the sequence $(||f_k||_p)_{k=1}^\infty$ rapidly decreases. The decomposition easily implies that $(T_{nk})_{n=0,k=1}^\infty$ is a basis in $\mathcal{E}_0(K)$.

However, the transition from $\mathcal{E}_0(K)$ to the space $\mathcal{E}(K)$ turned out to be difficult. Only in [40] an approach was found based on the following convolution property for the coefficients of scaling Chebyshev polynomials.

Fix in an arbitrary way three (maybe overlapping) finite intervals $I_1, I_2, I_3 \subset \mathbb{R}$. Denote by $\widetilde{T_{ni}}$, $i = 1, 2, 3$, $n \in \mathbb{Z}_+$, the corresponding scaling Chebyshev polynomial considered on the whole line. Fix $p, r \in \mathbb{Z}_+$ with $p \le r$. Then

$$\sum_{q=p}^{r} \xi_{p3}(\widetilde{T_{q2}})\, \xi_{q2}(\widetilde{T_{r1}}) = \xi_{p3}(\widetilde{T_{r1}}). \tag{8.1}$$

This property follows from the arguments of linear algebra. Let $(e_{i1})_{i=1}^n$, $(e_{i2})_{i=1}^n$, $(e_{i3})_{i=1}^n$ be bases in an n-dimensional vector space, $\alpha_{ik}(e_{jl})$ be the i-th coefficient in the expansion of e_{jl} in the k-th basis. Then

$$\sum_{q=1}^{n} \alpha_{p3}(e_{q2})\, \alpha_{q2}(e_{r1}) = \alpha_{p3}(e_{r1}). \tag{8.2}$$

The numbers $\alpha_{j3}(e_{q2})$ form the transition matrix $M_{3\leftarrow 2}$ from the second basis to the third. Here j is the index of row, q is the index of column. Analogously, $M_{2\leftarrow 1} = [\alpha_{i2}(e_{r1})]_{i,r=1}^n$. Thus in the sum above we see a product of the p-th row of $M_{3\leftarrow 2}$ with the r-th column of $M_{2\leftarrow 1}$ that is the (p, r)-th element of $M_{3\leftarrow 2}M_{2\leftarrow 1} = M_{3\leftarrow 1}$.

Now if we apply this argument to the Chebyshev bases (or arbitrary other polynomials of increasing degree) in $\mathcal{P}_r$, then the terms with $q < p$ vanish due to the orthogonality of ξ_{qk} to all polynomials of degree less than q.

The property (8.1) allows us to introduce a new biorthogonal system which will be a basis in $\mathcal{E}(K)$. A non-decreasing function $l : \mathbb{N} \to \mathbb{Z}_+$ can be defined that decompose basis elements in two groups: if $n \ge l(k)$, then $e_{nk} = T_{nk}$, otherwise $e_{nk} = \widetilde{T_{nk}}$ restricted to the set $[0, b_k] \bigcap K$. Respectively,

$\eta_{nk} = \xi_{nk}$ for $n \geq l(k)$, otherwise

$$\eta_{nk} = \xi_{nk} - \sum_{i=n}^{l(k-1)-1} \xi_{nk}(e_{i,\,k-1})\,\xi_{i,\,k-1}. \tag{8.3}$$

The system of functionals $(\eta_{nk})_{n=0,k=1}^{\infty}$ is total on $\mathcal{E}(K)$ and biorthogonal to $(e_{nk})_{n=0,k=1}^{\infty}$.

By [37], for compact sets $K = \{0\} \bigcup \bigcup_{k=1}^{\infty} I_k$, the condition

$$\text{there exists} \quad M \quad \text{so that} \quad |I_k| \geq dist(I_k, I_{k+1})^M \quad \text{for all} \quad k \tag{8.4}$$

determines "non-sharp cusps", when $\mathcal{E}(K) \simeq s$. For both cases (with or without the above condition) a basis in $\mathcal{E}(K)$ was constructed in [40]. It is possible to choose $(l(k))_{k=1}^{\infty}$ (depending on K) so that (2.1) is valid, therefore $(e_{nk})_{n=0,k=1}^{\infty}$ is a basis in $\mathcal{E}(K)$.

This approach was used for the construction of a special basis in the space $C^\infty[0,1]$ and subsequently for the space of C^∞-functions on a graduated sharp cusp ([42]). First we fix K satisfying (8.4) and the above basis $(e_{nk})_{n=0,k=1}^{\infty}$ in $\mathcal{E}(K)$. Then using function u_δ given after (5.2), we extend e_{nk} in both directions of I_k for $n \geq l(k)$ and only to the right side of $[0, b_k]$ if $n < l(k)$. In both cases, the values $\delta = \delta_{n,k}$ are carefully chosen so that the projection

$$Q : C^\infty[0,1] \to C^\infty[0,1] : F \mapsto \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \eta_{nk}(f) \cdot \tilde{e}_{nk}$$

is continuous. Here, $f = F|_K$ and $\tilde{e}_{nk}$ is the above extension of e_{nk}. This gives the decomposition $C^\infty[0,1] = X_1 \oplus X_0$, where $X_1 = Q(C^\infty[0,1])$ has the basis $(\tilde{e}_{nk})_{n=0,k=1}^{\infty}$ and $X_0 = \{F \in C^\infty[0,1] : \operatorname{supp} F \subset \bigcup_{k=2}^{\infty} [b_k, a_{k-1}]\} = (\oplus_{k=2}^{\infty} Q_k(C^\infty[0,1]))_s$. The projection Q_k is defined as $Q_k(F) = F - Q(F)$ restricted to $[b_k, a_{k-1}]$ and 0 otherwise on $[0,1]$. By $C_0^\infty[a,b]$ we denote the space of $C^\infty-$functions vanishing with all derivatives at the endpoints of the interval $[a,b]$. Since $Q_k(C^\infty[0,1]) \simeq C_0^\infty[b_k, a_{k-1}]$, it remains to find bases in each space of this kind. The Hermite functions $h_n, n \in \mathbb{Z}_+$, form a basis in the space S of rapidly decreasing on the line functions. Following [79], L.27, we use the scaling (for the interval $[b_k, a_{k-1}]$) functions $(h_{nk})_{n=0,k=2}^{\infty}$. Extending them by zero outside the interval give $(\tilde{h}_{nk})_{n=0,k=2}^{\infty} \subset C^\infty[0,1]$. The functions $\tilde{e}_{nk}, \tilde{h}_{n,k+1}$, $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$ form a basis in the space $C^\infty[0,1]$.

The task is now to present a basis in the space $C^\infty(\bar{\Omega}_\psi)$ for the following domain of the form of graduated sharp cusp with arbitrary sharpness of the spike. Let us fix a sequence $(\psi_k)_{k=1}^{\infty}$ where $\psi_k \searrow 0$ as fast as we wish and consider the step-function $\psi : \psi(x) = \psi_k$ if $a_k \leq x < a_{k-1}$, $k \in \mathbb{N}$. The points a_k are the left endpoints of the intervals I_k from the set K chosen above. Let $\Omega_\psi = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, |y| < \psi(x)\}$. The compact set $K = \bar{\Omega}_\psi$ is Whitney regular, so $C^\infty(\bar{\Omega}_\psi) \simeq \mathcal{E}(K)$.

Let us denote by R_k the rectangle $I_k \times [-\psi_k, \psi_k]$, by R'_k the rectangle $[b_k, a_{k-1}] \times [-\psi_k, \psi_k]$. Let $E_1 = \{0\} \cup \bigcup_{k=1}^{\infty} R_k$. Then the functions $e_{nmk}(x,y) = e_{nk}(x) T_m(\frac{y}{\psi_k})\Big|_K$, $n, m \in \mathbb{Z}_+$, $k \in \mathbb{N}$, form a basis in $\mathcal{E}(E_1)$. As above, a sequence $(l(k))_{k=1}^{\infty}$ can be chosen that will determine the type of biorogonal functional: if $n \geq l(k)$ then

$$\eta_{nmk}(f) = \xi_{nmk}(f) := \frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x_k + \delta_k \cos t, \psi_k \cos\tau) \cos nt \cos m\tau\, dt d\tau.$$

Otherwise, for small n, the functional η_{nmk} is a multi-variable version of (8.3).

The rest of the proof runs as in the construction of a special basis. We extend e_{nmk} as follows $\tilde{e}_{nmk}(x,y) = \tilde{e}_{nk}(x) T_m(\frac{y}{\psi_k})$ for $(x,y) \in \Omega_\psi$. Then we show that the projection

$$Q : C^\infty(\bar{\Omega}_\psi) \to C^\infty(\bar{\Omega}_\psi) : F \mapsto \sum_{k=1}^\infty \sum_{n=0}^\infty \sum_{m=0}^\infty \eta_{nmk}(f) \cdot \tilde{e}_{nmk}$$

is well defined and continuous. In a similar as above fashion, $C^\infty(\bar{\Omega}_\psi) = Y_1 \oplus Y_0$ with $Y_1 = Q(C^\infty(\bar{\Omega}_\psi)), Y_0 = (\oplus_{k=2}^\infty Q_k(C^\infty(\bar{\Omega}_\psi)))_s$, where the projection Q_k is $Id - Q$ on R'_k and 0 otherwise on $\bar{\Omega}_\psi$. Now

$$Q_k(C^\infty(\bar{\Omega}_\psi)) = \{F \in C^\infty(\bar{\Omega}_\psi) : \operatorname{supp} F \subset R'_k\} = C_0^\infty[b_k, a_{k-1}] \tilde{\otimes}_\pi C^\infty[-\psi_k, \psi_k].$$

The functions $(\tilde{h}_{nmk})_{n,m=0}^\infty$, where $\tilde{h}_{nmk}(x,y) = \tilde{h}_{nk}(x) T_m(\frac{y}{\psi_k})$, form a basis in the subspace $Q_k(C^\infty(\bar{\Omega}_\psi))$. Therefore the functions $\tilde{e}_{nmk}, \tilde{h}_{n,m,k+1},\ n,m \in \mathbb{Z}_+,\ k \in \mathbb{N}$ form a basis in the space $C^\infty(\bar{\Omega}_\psi)$.

9 Method of Local Interpolations

The functional (8.3) was obtained by means of the orthogonalization in the dual space. This approach can be used also in other cases where there is no biorthogonality of original systems. Let us present functionals that correspond to the local Newton interpolation of functions.

First we consider the formula for convolution property (8.2) corresponding to this case. Suppose we are given three sequences $(x_k^{(s)})_{k=1}^\infty$, $s = 1,2,3$, such that for a fixed superscript s all points in the sequence $(x_k^{(s)})_{k=0}^\infty$ are distinct. As in Section 7, let $e_{0s} \equiv 1$, $e_{ns} = \prod_{k=1}^n (x - x_k^{(s)})$ and $\xi_{ns}(f) = [x_1^{(s)}, x_2^{(s)}, \cdots x_{n+1}^{(s)}]f$ for $n \in \mathbb{Z}_+, f \in \mathcal{X}$. Here $\mathcal{X}$ a Fréchet space of functions on a set containing all points $x_k^{(s)}$. Then by (8.2),

$$\sum_{q=p}^r \xi_{p3}(e_{q2})\, \xi_{q2}(e_{r1}) = \xi_{p3}(e_{r1}) \qquad \text{for} \quad p \le r. \tag{9.1}$$

We now turn to the case of a chain of compact sets $K_0 \supset K_1 \supset \cdots \supset K_s \supset \cdots$ and finite systems of distinct points $(x_k^{(s)})_{k=1}^{N_s} \subset K_s$ for $s = 0,1,\cdots$. Some part of the interpolation nodes on K_{s+1} - let $(x_k^{(s+1)})_{k=1}^{M_{s+1}}-$ belongs to the previous set $(x_k^{(s)})_{k=1}^{N_s}$. Of course, $M_{s+1} \le \min\{N_s, N_{s+1}\}$. For any $s \ge 0$ and $n = M_s+1, \cdots, N_s$, we define e_{ns} as above for $x \in K_s$ and $e_{ns} = 0$ for $x \in K_0 \setminus K_s$. Let $K_{s-1} \setminus K_s$ be closed for any $s \ge 1$. Then the functions e_{ns} are continuous on K_0. Similarly, $\xi_{ns}(f) = [x_1^{(s)}, x_2^{(s)}, \cdots x_{n+1}^{(s)}]f$ with $x_{N_s+1}^{(s)} := x_{M_{s+1}+1}^{(s+1)}$. We see at once that $\xi_{ns}(e_{m,\,s+1}) = 0$, because the number $\xi_{ns}(f)$ is defined by values of f at some points on $K_s \setminus K_{s+1}$ and at some $(x_k^{(s+1)})_{k=1}^{M_{s+1}}$, where the function $e_{m,\,s+1}$ is zero. Clearly, $\xi_{n,\,s+1}(e_{ms}) = 0$ for $n > m$. But for $n \le m$ the functional $\xi_{n,\,s+1}$ in general is not biorthogonal to e_{ms}. For this reason, we subtract from $\xi_{n,\,s+1}$ its projection (in the dual space) onto the subspace spanned by the functionals corresponding to such elements. This gives the functional

$$\eta_{n,\,s+1} = \xi_{n,\,s+1} - \sum_{k=n}^{N_s} \xi_{n,\,s+1}(e_{ks})\, \xi_{ks}. \tag{9.2}$$

It is biorthogonal not only to the elements e_{ms} of the $s-$th level, as is easy to see, but also, by (9.1), to all e_{mj} with $j = 0,1,\cdots,s-1$.

Thus, at the $s-$th step of the above procedure, we interpolate the given function f at points of K_s, then omit a part of nodes and continue to interpolate f at some new points from K_{s+1}. On the one hand, we can control degrees of interpolating polynomials. On the other hand, the diameters of sets decrease. This improves the approximation properties of the interpolating polynomials.

The functional $\eta_{n,\,s+1}(f) = [x_1^{(s+1)}, x_2^{(s+1)}, \cdots x_{n+1}^{(s+1)}]f-$

$$\sum_{k=n}^{N_s} [x_1^{(s+1)}, x_2^{(s+1)}, \cdots x_{n+1}^{(s+1)}] \left(\prod_{i=n}^{k} (x - x_i^{(s)}) \right) \cdot [x_1^{(s)}, x_2^{(s+1)}, \cdots x_{k+1}^{(s)}]f$$

plays a role of the divided difference in local interpolations.

Since the points of interpolation are chosen independently on functions and Newton's way to interpolate assumes nested families of the nodes, this approach allows us to construct in the next section topological bases in spaces of functions defined on rarefied sets.

The case of local Taylor interpolation was considered in [48]. Let $K = K(\mathcal{L})$ be, as in Section 6, the Cantor set associated with a sequence $\mathcal{L} = (\ell_s)_{s=0}^{\infty}$, so $K = \bigcap_{s=0}^{\infty} \bigcup_{j=1}^{2^s} I_{j,s}$ where $I_{j,s} = [a_{j,s}, b_{j,s}]$ with $b_{j,s} - a_{j,s} = \ell_s$. A basis in the Banach space $\mathcal{E}^p(K)$ for a fixed p was constructed.

For $s \in \mathbb{Z}_+, 0 \le k \le p$ and $1 \le j \le 2^s$, let $e_{k,j,s}(x) = (x - a_{j,s})^k/k!$ if $x \in K \bigcap I_{j,s}$ and $e_{k,j,\,s} = 0$ on K otherwise. Given $f \in \mathcal{E}^p(K)$ let $\xi_{k,j,\,s}(f) = f^{(k)}(a_{j,s})$ for the same values of s, j, and k as above.

For a fixed level s, the system $(e_{k,j,\,s}, \xi_{k,j,\,s})$ is biorthogonal. In order to obtain biorthogonality as well with regard to s, we use (8.1), which now for $I_{i,\,n} \supset I_{j,\,s-1}$ is as follows

$$\sum_{m=k}^{q} \xi_{k,2j,\,s}(e_{m,j,\,s-1}) \cdot \xi_{m,j,\,s-1}(e_{q,i,\,n}) = \xi_{k,2j,\,s}(e_{q,i,\,n}) \quad \text{for all} \quad q \le p.$$

Now this identity is the following simple binomial expansion:

$$\sum_{m=k}^{q} \frac{(a_{2j,s} - a_{j,s-1})^{m-k}}{(m-k)!} \cdot \frac{(a_{j,s-1} - a_{i,n})^{q-m}}{(q-m)!} = \frac{(a_{2j,s} - a_{i,n})^{q-k}}{(q-k)!}.$$

Here we consider summation until q since for $q < m \le p$, the terms $\xi_{m,j,\,s-1}(e_{q,i,\,n})$ vanish.

10 Bases in $\mathcal{E}(K)$ for Cantor-Type Sets

Since we plan to use function interpolation on a compact set K, the choice of interpolating nodes is critical. Of course, our goal is to minimize the norms of interpolation projections (Lebesgue constants). A good choice here are Fekete points $(x_{k,n})_{k=1}^n$, they maximize $\prod_{0 \le i < j \le n} |x_{i,n} - x_{j,n}|$ which is the modulus of corresponding Vandermonde determinant. These points were used in the construction of operator (5.2). It's easy to see that $\Lambda_{n-1}(K, (x_{k,n})_{k=1}^n) \le n$ for each compact set K. However, for Newton interpolation with nested sets of interpolating nodes, the only sequence $(x_n)_1^\infty$ needs to be chosen. A nested analog of Fekete array is given by Leja sequnce: the points $(x_n)_{n=1}^\infty \subset K$ are Leja if $x_1 \in K$ is arbitrary, and, once $x_1, x_2, \ldots, x_{k-1}$ have been determined, x_k is chosen so that it provides the maximum modulus of the polynomial $(x - x_1) \cdots (x - x_{k-1})$ on K. It should be noted that the exact location of the Fekete points is known only for a few compact sets. The situation with Leja sequence is even worse. For this reason, in specific questions related to interpolation, approximate Fekete (Leja) points are usually considered. Our object is small sets of Cantor type $K(\mathcal{L})$. For them, the natural choice for "almost" Leja points are the endpoints of intervals in Cantor procedure, arranged in a reasonable way. For this purpose, in a number of articles the author used *the rule of increase of type*. Let us describe it.

Let x be an endpoint of some basic interval $I_{j,s}$. Then there exists a minimal number s (the *type* of x) such that x is the endpoint of some $I_{j,m}$ for every $m \geq s$. By X_k we denote all endpoints of type k. Hence, $X_0 := \{0, 1\}, X_1 := \{\ell_1, 1 - \ell_1\}, X_2 := \{\ell_2, \ell_1 - \ell_2, 1 - \ell_1 + \ell_2, 1 - \ell_2\}$, etc. The set $Y_s = \bigcup_{k=0}^{s} X_k$ contains 2^{s+1} points. To arrange all x from $\cup_{k=0}^{\infty} X_k$ in order, first we take the points from $X_0 \cup X_1$ as follows: $x_1 = 0, x_2 = 1, x_3 = \ell_1, x_4 = 1 - \ell_1$. To put the points from X_2 in order we increasingly arrange x from Y_1, so $Y_1 = \{x_1, x_3, x_4, x_2\}$. After this we increase the index of each point by 4. This gives the ordering $X_2 = \{x_5, x_7, x_8, x_6\}$. Similarly, indices of increasingly arranged points from $Y_{k-1} = \{x_{i_1}, x_{i_2}, \cdots, x_{i_{2^k}}\}$ define the ordering $X_k = \{x_{i_1+2^k}, x_{i_2+2^k}, \cdots, x_{i_{2^k}+2^k}\}$. We see that $x_{j+2^k} = x_j \pm \ell_k$, where the sign is uniquely defined by j. The exact location of x_n in terms of ℓ_i is determined by the binomial expansion of n.

Thus we have fixed the sequence $(x_n)_{n=1}^{\infty} \subset K(\mathcal{L})$. As above, we take $e_0 \equiv 1, e_n(x) = \prod_1^n (x - x_k)$ for $n \in \mathbb{N}$ and $\xi_n(f) = [x_1, x_2, \cdots, x_{n+1}]f$ for $n \in \mathbb{Z}_+, f \in \mathcal{E}(K(\mathcal{L}))$. The biorthogonal system $(e_n, \xi_n)_{n=0}^{\infty}$ is a basis in $\mathcal{E}(K(\mathcal{L}))$ for sufficiently small sets $K(\mathcal{L})$. In [44], the sets $K^{(\alpha)}$ given after Example 6.2 were considered, so $\mathcal{L} = \{\ell_0 = 1, \ell_1, \ell_2 = \ell_1^{\alpha_2}, \cdots, \ell_s = \ell_1^{\alpha_2\alpha_3\cdots\alpha_s}, \cdots\}$. Here we assume that the exponents α_s for $s \geq 2$ are such that $0 < 3\ell_{s+1} \leq \ell_s$ for each s. By Theorem 1 in [44], if $\alpha_s \geq 2$ for all s then $(e_n, \xi_n)_{n=0}^{\infty}$ is a basis in $\mathcal{E}(K^{(\alpha)})$. Since the Markov factors for such sets have very fast growth, the argument of Theorem 7.2 cannot be used. By direct estimation of norms, (1.2) was proved for $(e_n, \xi_n)_{n=0}^{\infty}$.

However, for $\alpha_s < 2$ the condition of Dynin-Mityagin criterion is not valid for this system, so the method of local interpolations is used. First we fix a non-decreasing sequence of natural numbers $(n_s)_{s=0}^{\infty}$. It determines the value $N_s = 2^{n_s}$, which is the maximum quantity of interpolating nodes $x_{n,j,s}$ chosen by the rule of increase of type on each basic interval $I_{j,s}$ for $1 \leq j \leq 2^s$. The corresponding functional $\xi_{N,j,s}$ is the divided difference at $N_s + 1$ points. According to the rule, the last point belongs to the left subinterval of $I_{j,s}$. Then we split these points into two parts containing $M_{s+1}^{(l)} = N_s/2 + 1$ points for the left subinterval $I_{2j-1,s+1}$ and, accordingly, $M_{s+1}^{(r)} = N_s/2$ for the right one. After this, we continue interpolation on these subintervals separately up to N_{s+1} points on each.

The basis elements of level zero are $e_{N,1,0} = \prod_{n=1}^{N}(x - x_{n,1,0}) = \prod_1^N (x - x_n)$ for $x \in K^{(\alpha)}$, $N = 0, 1, \cdots, N_0$. Then for $s \geq 1$ and $1 \leq j \leq 2^s$ let us take $e_{N,j,s} = \prod_{n=1}^{N}(x - x_{n,j,s})$ if $x \in K^{(\alpha)} \cap I_{j,s}$ and $e_{N,j,s} = 0$ on $K^{(\alpha)}$ otherwise. Here, $N = M_s^{(a)}, M_s^{(a)} + 1, \cdots, N_s$ with $a = l$ for odd j and $a = r$ if j is even. Biorthogonal functionals $\eta_{N,j,s}$ are given as in (9.2).

This determines the biorthogonal system $(e, \eta) := (e_{N,j,s}, \eta_{N,j,s})_{s=0,\, j=1,\, N=M_s}^{\infty,\; 2^s,\; N_s}$ with the total sequence of functionals. By Theorem 2 in [44], (e, η) satisfies (1.2) provided that a suitable choice of $(n_s)_0^{\infty}$ is made, namely, the sequence $(2^{N_s} l_s^Q)_{s=0}^{\infty}$ is bounded for some Q.

In particular, (e, η) is a basis in $\mathcal{E}(K_0)$ for $n_s = [\log_2 s]$ for $s \geq 2$. Here K_0 is the classical Cantor set and $[\,b\,]$ denotes the greatest integer in b. Of course, different proper choices of $(n_s)_0^{\infty}$ give a variety of different bases in each space $\mathcal{E}(K^{(\alpha)})$. All these bases are not polynomial, whereas the above basis in $\mathcal{E}(K^{(\alpha)})$ for $\alpha_s \geq 2$ is strict polynomial basis. The question arises about the universality of Faber bases in the spaces $\mathcal{E}(K)$. In [53], it was shown that for each compact subset K of $\mathbb{R}$ of infinite cardinality with an isolated point, the space of Whitney jets $\mathcal{E}(K)$ does not possess a basis consisting only of polynomials. On the other hand, Faber interpolating bases were presented in two extreme (in the sense of density) cases, for $[-1, 1]$ and $K^{(\alpha)}$ with large (α). The common feature of these sets is that we know "almost" Leja points for them. The author believes that for $K^{(\alpha)}$ with small (α), the proposed choice of interpolating points is far from optimal. This is the reason for the absence of the basic property for $(e_n, \xi_n)_{n=0}^{\infty}$ in this case. Moreover, in [53] the following hypothesis was proposed.

Conjecture 10.1. *Given a compact set $K \subset \mathbb{R}$ of infinite cardinality, the space $\mathcal{E}(K)$ has a polynomial basis if and only if the set K is perfect. In addition, if K is perfect, then $\mathcal{E}(K)$ possesses a strict polynomial basis.*

Similarly, in [52], bases for the space $\mathcal{E}(K(\gamma)$, were presented that are strictly polynomial for sufficiently small γ, which means that the set $K(\gamma)$ is small, and, by local interpolations for each γ.

11 Bases in the Spaces of Witney Jets and Restriction Spaces

As mentioned above, by [53], if $K \subset \mathbb{R}$ contains infinitely many points and at least one of them is an isolated point, then $\mathcal{E}(K)$ cannot have a basis consisting only of polynomials. It's easy to see that polynomials are dense in any Whitney space. Combining these results does not mean that $\mathcal{E}(K)$ has no basis at all, as we see in the following trivial example from [53].

Example 11.1. *Let $K = [-1, 1] \bigcup \{2\}$. Then $\mathcal{E}(K) = \mathcal{E}([-1, 1]) \oplus \mathcal{E}(\{2\})$. We know that the Chebyshev polynomials $(T_n)_{n=0}^{\infty}$ form a basis in the space $\mathcal{E}([-1, 1])$. For a basis in the space $\mathcal{E}(\{2\})$ we take the jets $(e_n)_{n=0}^{\infty}$ given by the functions $(x-2)^n/n!$ at $x = 2$, $n \in \mathbb{Z}_+$. Let $f_{2n-1}(x) = T_{n-1}(x)$ for $|x| \leq 1$, $f_{2n-1}^{(j)}(2) = 0$ and $f_{2n} = e_{n-1}$ for $n \in \mathbb{N}, j \in \mathbb{Z}_+$. Then $(f_n)_{n=1}^{\infty}$ is a basis in $\mathcal{E}(K)$.*

The correct conclusion from the above results is that generally bases in Fréchet spaces are possibly unstable. With a basis in Banach spaces the situation is different: a small perturbation of the basis elements preserves the basis property of the system.

These considerations raise interest in the question of the existence of a basis in the space $\mathcal{E}(K)$ of jets defined on a compact set that contains infinitely many isolated points. Of course, the case of N isolated points is trivial since then $\mathcal{E}(K) \simeq \omega^N$.

In [54], the set $K_\varphi = \{0\} \bigcup \bigcup_{k=0}^{\infty} \{a_k\}$ was considered, where $a_k = e^{-\varphi(k)}$ for φ from the class Φ of functions on $[0, \infty)$ satisfying certain mild regularity conditions, see [54] or [57] for more details. We write Φ_{sub} for the subclass of functions with sub-exponential growth rate, which means that $\lim_{x\to\infty} \frac{\log \varphi(x)}{x} = 0$.

First, for given $n, k \in \mathbb{Z}_+$, let e_{nk} be a jet defined by the function $(x - a_k)^n/n!$ restricted to $x = a_k$, so $e_{nk}^{(j)}(x) = 1$ if $j = n$ and $x = a_k$ and $e_{nk}^{(j)}(x) = 0$ in all other cases. Thus $(e_{nk})_{n=0}^{\infty}$ is a basis in $\mathcal{E}(\{a_k\})$. Let us introduce a system $(f_{nk})_{n=0,k=0}^{\infty}$ which is a basis $\mathcal{E}(K_\varphi)$. It consist of elements of two types. Let $f_{nk} = e_{nk}$ for $n \geq 1$. The jets $(f_{nk})_{n=1}^{\infty}$ represent derivatives of f at the point a_k for $k \in \mathbb{Z}_+$.

To present the function values, we fix a sequence $(k_s)_{s=1}^{\infty}$ of natural numbers with (possibly not strictly) $k_{s+1} - k_s \nearrow \infty$. By the method of local interpolations, given $f \in \mathcal{E}(K_\varphi)$, we interpolate f at $a_{k_1}, \cdots, a_{k_3}$. After this, we omit a part of nodes and, for $k_3 < k \leq k_4$, represent $f(a_k)$ as $L_{k-k_2}(f, a_k; (a_j)_{j=k_2}^{k})$. Continuing in this way, at the $s-$th step with $s \geq 2$, for $k_{s+1} < k \leq k_{s+2}$ we present $f(a_k)$ as the value of interpolating polynomial with nodes $(a_j)_{j=k_s}^{k}$. It follows that f_{0k} must be a polynomial on $[0, a_{k_s}]$ with zeros at $a_{k_s}, \ldots, a_{k-1}$.

Let integer numbers M, N with $0 \leq M \leq N$ be given. Set $n = N - M + 1$. Then the $n-$th degree polynomial $\Omega_{M,N}(x) = \prod_{k=M}^{N}(x - a_k)$ determines a jet $\omega_{M,N} = \Omega_{M,N} \cdot \chi_{[0,a_M]}$ from $\mathcal{E}(K_\varphi)$, where χ_I is the characteristic function of an interval I. Now the jets f_{0k} are defined as follows. If $0 \leq k < k_1$ then let $f_{0k} = e_{0k}$. If $k_1 \leq k \leq k_3$ then $f_{0k} = \omega_{k_1,k-1}$. If $k_{s+1} < k \leq k_{s+2}$ for $s \geq 2$ then $f_{0k} = \omega_{k_s,k-1}$. Correspondingly, $\zeta_{0k}(f) = f(a_k)$ for $k < k_1$, $\zeta_{0k}(f) = \xi_{k_1,k}(f)$ for $k_1 \leq k \leq k_3$ and, if $k_{s+1} < k \leq k_{s+2}$ for $s \geq 2$ then following (9.2), the functional corresponding to f_{0k} is

$$\zeta_{0,k}(f) = \xi_{k_s,k}(f) - \sum_{j=0}^{k_{s+1}} \xi_{k_s,k}(\omega_{k_{s-1},j-1}) \cdot \xi_{k_{s-1},j}(f).$$

Indeed, the real summation here only occurs from $j = k - k_s + k_{s-1}$, since the previous terms have disappeared due to biorthogonality.

Similarly, for $n \ge 1$ and $k_r < k \le k_{r+1}$ with $\eta_{nk}(f) = f^{(n)}(a_k)$ we define

$$\zeta_{n,k}(f) = \eta_{nk}(f) - \sum_{j=0}^{k_{r+2}} \eta_{nk}(f_{0j})\zeta_{0,j}(f).$$

Then the system $(f,\zeta) := (f_{nk},\zeta_{nk})_{n,k=0}^{\infty}$ is biorthogonal. The functionals $(\zeta_{nk})_{n,k=0}^{\infty}$ are total over $\mathcal{E}(K_\varphi)$. By Theorem 8.1 in [54], $(f_{nk})_{n,k=0}^{\infty}$ is a basis in $\mathcal{E}(K_\varphi)$ for $\varphi \in \Phi_{\text{sub}}$.

We now turn to the class of restriction spaces. Instead of the short exact sequence (4.7), let us consider

$$0 \longrightarrow \mathcal{Z}(K,B) \xrightarrow{i} C^\infty(B) \xrightarrow{\pi} C_\infty(K) \longrightarrow 0,$$

which corresponds to the ideal of zero on K functions $\mathcal{Z}(K,B) = \{F \in C^\infty(B) : F|_K = 0\}$ and the restriction space $C_\infty(K) = \{F|_K : F \in C^\infty(B)\}$.

As noted at the end of Section 4, the description of functions from $C_\infty(K)$ for the multidimensional case is quite difficult.

Even so, for $K \subset \mathbb{R}$, thanks to Whitney [117], there is a simple condition for a function on K to belong to $C_\infty(K)$, namely, its divided differences of any order are uniformly continuous in the following sense: a function f defined on K belongs to $C_\infty(K)$ if and only if for arbitrary $p \in \mathbb{N}$ and any $\varepsilon > 0$ there is a positive δ such that for each $x \in K$ and any pair of sets of $p+1$ points $(x_k)_{j=0}^p, (x'_k)_{j=0}^p \subset K \bigcap (x-\delta, x-\delta)$ (the points of each set are distinct) we have

$$|[x_0,\dots,x_p]f - [x'_0,\dots,x'_p]f| < \varepsilon.$$

It follows that the topology of $C_\infty(K)$ can be given by the norms

$$|f|_p = \sup\{|[x_0,\dots,x_j]f| : 0 \le j \le p\},$$

where $x_0,\dots,x_j$ are distinct points in $K, p \in \mathbb{Z}_+$.

The ideal $\mathcal{Z}_W(K) = \{f \in \mathcal{E}(K) : f|_K = 0\}$ of the algebra $\mathcal{E}(K)$ determines the following short exact sequence

$$0 \longrightarrow \mathcal{Z}_W(K) \xrightarrow{i} \mathcal{E}(K) \xrightarrow{\pi} C_\infty(K) \longrightarrow 0.$$

We are interested in the problem of splitting for

$$0 \longrightarrow \mathcal{Z}(K_\varphi,[0,1]) \xrightarrow{i} C^\infty[0,1] \xrightarrow{\pi} C_\infty(K_\varphi) \longrightarrow 0, \tag{11.1}$$

and

$$0 \longrightarrow \mathcal{Z}_W(K_\varphi) \xrightarrow{i} \mathcal{E}(K_\varphi) \xrightarrow{\pi} C_\infty(K_\varphi) \longrightarrow 0. \tag{11.2}$$

It is easy to show that functions from $\mathcal{Z}_W(K_\varphi) = \{f \in \mathcal{E}(K_\varphi) : f(a_k) = 0 \text{ for } k \in \mathbb{Z}_+\}$ are flat at the origin, that is $f^{(n)}(0) = 0$ for each $n \in \mathbb{Z}_+$. In the case of $\varphi \in \Phi_{\text{sub}}$, $\mathcal{E}(K_\varphi)$ we have immediately $\mathcal{Z}_W(K_\varphi) = \overline{span\{(f_{nk})_{n=1,k=0}^{\infty}\}}$. This can be extended to the whole class:

Lemma 18 in [57]: *$(f_{nk})_{n=1,k=0}^{\infty}$ is a basis in the space $\mathcal{Z}_W(K_\varphi)$ for each $\varphi \in \Phi$.*

Of course, function counterparts of the jets $f_{0k}, k \in \mathbb{Z}_+$, are the first candidates for the basis in $C_\infty(K_\varphi)$. The jet $\omega_{M,N}$ determines a function $g_{M,N}$ from $C_\infty(K)$. Thus, $g_{M,N}(a_j) = 0$ for $j \le N$ and $g_{M,N}(x) = \prod_{k=M}^N (x-a_k)$ for $x = 0$ or $x = a_j$ with $j \ge N+1$. Let g_k be a function counterpart of f_{0k} for $k \in \mathbb{Z}_+$.

Theorem 2 in [57]: *$(g_k)_{k=0}^{\infty}$ is a basis in the space $C_\infty(K)$ for each $\varphi \in \Phi$.*

Clearly, $|g_k|_p \le C_p\,||f_{0k}||_p$ for all k with some constant C_p. If $\varphi \in \Phi_{\rm sub}$ then the inverse inequality is also valid. However, $||f_{0k}||_p$ are essentially larger than $|g_k|_p$ for $\varphi \notin \Phi_{\rm sub}$.

Now it is easy to check ([57], T.3) that $C_\infty(K_\varphi)$ has a dominating norm if and only if $\varphi \in \Phi_{\rm sub}$. Thus, by Vogt's splitting theory (Section 13), (11.1) splits if and only if $\varphi \in \Phi_{\rm sub}$. Also (11.2) splits for $\varphi \in \Phi_{\rm sub}$.

In the case $\varphi \notin \Phi_{\rm sub}$ situation is more complicated. Here there are bases in both utmost spaces for the sequence (11.2), but the problems of existence of a basis in the middle space and its splitting remain open.

12 Counting Linear Topological Invariants

As mentioned in Section 2, the coordinate form of spaces with bases simplifies the use of linear topological invariants (LTI). Let us consider those LTI that can be utilized to distinguish the spaces under analysis. Let $\tilde{\mathcal{X}}$ be a class of locally convex spaces and Θ a set with a relation of equivalence $\sim$. Suppose $\theta : \tilde{\mathcal{X}} \to \Theta$ is a mapping so that $\mathcal{X} \simeq \mathcal{Y} \Longrightarrow \theta(\mathcal{X}) \sim \theta(\mathcal{Y})$. Then we say that Θ is a linear topological invariant on $\tilde{\mathcal{X}}$. It is complete in $\tilde{\mathcal{X}}$ if the inverse to above implication is true.

Invariants such as separability or existence of a basis are common to both B and NF classes of spaces. However, typical LTI for NF-spaces do not apply to B-spaces and vice versa. Important topological properties of B-spaces are determined by the shape of its unit ball, the nature of its convexity. On the other hand, each NF-space $\mathcal{X}$ is countably Hilbert, so a base of zero neighborhoods $(U_p)_{p=1}^\infty$ for $\mathcal{X}$ can be chosen from ellipsoids (if we define one of its neighborhood as a ball) with the best possible type of convexity. As in Section 1, we equip a Fréchet space $\mathcal{X}$ with an increasing seminorms $(\|\cdot\|_p)_{p=0}^\infty$. Set $U_p = \{f \in \mathcal{X} : \|f\|_p \le 1.\}$

Another important difference between B and NF classes of spaces is that each NF-space is a *Schwartz* space, which means that for each p there is q so that U_q is precompact in the normed space $(\mathcal{X}, |\cdot|_p)$. Hence the number

$$N(U_q, \varepsilon U_p) = \min\left\{N : U_q \subset \bigcup_{k=1}^N (x_k + \varepsilon U_p) : x_1, \ldots, x_N \in \mathcal{X}\right\}$$

is well-defined. Its logarithm is called the $\varepsilon-$entropy of U_q with respect to U_p. Kolmogorov ([67]) suggested to use $\varepsilon-$entropy to distinguish Schwartz spaces. Any such space determines a class of functions

$$\Psi_{\mathcal{X}} = \{\psi : (0,\infty) \longrightarrow \mathbb{R}_+ : \forall p\ \exists q, \varepsilon_0 : N(U_q, \varepsilon U_p) \le \psi(\varepsilon) \text{ for all } \ 0 < \varepsilon < \varepsilon_0\}.$$

Two Schwartz spaces $\mathcal{X}$ and $\mathcal{Y}$ have the same *approximative dimension* if $\Psi_{\mathcal{X}} = \Psi_{\mathcal{Y}}$.

In particular, by means of this invariant it was shown that $A(\mathbb{D}^n) \not\simeq A(\mathbb{D}^m)$ for $m \ne n$. Here, $A(D)$ is the space of analytic functions in the domain D with the topology of uniform convergence on compact subsets of D, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

However, the calculation of approximate dimension is quite difficult. There is a similar but simpler approach suggested by Bessaga, Pełczyński, Rolewicz in [11]. It is based on the calculation of asymptotic for Kolmogorov diameters.

The $n-$th *Kolomogorov diameter* of U_q with respect to U_p is

$$d_n(U_q, U_p) = \inf_{L \in \mathcal{L}_n} \inf\{\delta : U_q \subset \delta U_p + L\}, \tag{12.1}$$

where $\mathcal{L}_n$ stands for n dimensional subspaces of $\mathcal{X}$. One can show that

$$d_n(U_q, U_p) = \inf_{L \in \mathcal{L}_n} \sup_{x \in U_q} \inf_{y \in L} ||x - y||_p.$$

Thus $d_n(U_q, U_p), n \in \mathbb{N}$, are semiaxes of the ellipsoid U_q with respect to the ball U_p.

The *diametral dimension* of a NF space X is a set of sequences

$$\Gamma(\mathcal{X}) := \{(\gamma_n)_{n=1}^\infty : \forall p \exists q : \gamma_n \cdot d_n(U_q, U_p) \to 0 \text{ as } n \to \infty\}.$$

A straightforward verification shows that $\mathcal{X} \simeq \mathcal{Y}$ implies $\Gamma(\mathcal{X}) = \Gamma(\mathcal{Y})$. It is especially easy to compute $\Gamma(\mathcal{X})$ when $\mathcal{X}$ is a NF-space with a regular basis. Following Dragilev [24], we say that a topological basis $(e_k)_{k=0}^\infty$ in a NF-space (or the corresponding Köthe matrix $(||e_k||_p)_{k,p=0}^\infty$) is *regular* if the sequence $(||e_k||_p/||e_k||_{p+1})_{k=0}^\infty$ decreases for each fixed p. In this case the subspace that realizes the first infimum in (12.1) is $\mathrm{span}(e_j)_{j=0}^{k-1}$, so $d_k(U_q, U_p) = ||e_k||_p/||e_k||_q$. In particular, $C^\infty[-1,1] \simeq s$, therefore

$$\Gamma(C^\infty[-1,1]) = \{(\gamma_n)_{n=1}^\infty : \forall p \exists q : \gamma_n \cdot n^{p-q} \to 0 \text{ as } n \to \infty\} = \{(\gamma_n)_{n=1}^\infty : \overline{\lim}_n \frac{\log|\gamma_n|}{\log n} < \infty\}.$$

Example 12.1. *The topology in the space of entire functions $A(\mathbb{C})$ is given by the norms $||f||_p = \sup_{|z|\le p} |f(z)|, p \in \mathbb{N}$, whereas in $A(\mathbb{D})$ by $||f||_p = \sup_{|z|\le 1-1/p} |f(z)|, p \ge 2$. The monomials $(z^k)_{k=0}^\infty$ form a basis in each space. Therefore, $A(\mathbb{D}) \simeq \mathcal{K}(\exp(-k/p))$ with*

$$\Gamma(A(\mathbb{D})) = \{(\gamma_n)_{n=1}^\infty : \forall \varepsilon \ \textit{ we have } \ \gamma_n \cdot (1-\varepsilon)^n \to 0 \ \textit{ as } \ n \to \infty\},$$

whereas $A(\mathbb{C}) \simeq \mathcal{K}(p^k)$ with

$$\Gamma(A(\mathbb{C})) = \{(\gamma_n)_{n=1}^\infty : \exists M : \gamma_n \cdot M^{-n} \to 0 \ \textit{ as } \ n \to \infty\}.$$

Hence, $A(\mathbb{D}) \not\simeq A(\mathbb{C})$.

The above spaces served as models for power series spaces. Suppose $\beta = (\beta_k)_{k=0}^\infty$ is an *exponent sequence*, that is strictly increasing to infinity sequence with $\beta_0 \ge 0$. Given $r \in (-\infty, +\infty]$, let $t_p \nearrow r$ as $p \to \infty$. A power series space is a Köthe space $\mathcal{K}(A)$ with $a_{k,p} = e^{t_p \beta_k}$. Depending on the finiteness of r, power series spaces of finite $E_r(\beta)$ or infinite $E_\infty(\beta)$ type were introduced. In both cases, they are Schwartz spaces. For their nuclearity, see for instance [78], 29.6. Since for a fixed β all $E_r(\beta)$ of finite type are isomorphic, the space $E_0(\beta)$ is usually considered. Thus, $A(\mathbb{C}) \simeq E_\infty(\beta)$ and $A(\mathbb{D}) \simeq E_0(\beta)$ for $\beta = (k)_{k=0}^\infty$. Also, $C^\infty[-1,1] \simeq E_\infty((\log k)_{k=1}^\infty)$.

As for the spaces under consideration, the diametral dimension cannot be applied to distinguish nuclear spaces $\mathcal{X}$ of the type $C^\infty(\overline{\Omega})$ or $\mathcal{E}(K)$ for K with nonempty interior. The reason is that these spaces contain a subspace which is isomorphic to s and thus their diametral dimension $\Gamma(\mathcal{X})$ is not larger than $\Gamma(s)$ (see for example [79] Proposition 7). On the other hand, by [79], the space s has the smallest diametral dimension in the class of nuclear spaces and we get $\Gamma(\mathcal{X}) = \Gamma(s)$.

Nevertheless, for K with $Int K = \emptyset$ the diametral dimension turned out to be quite effective. In two papers, [5] and [43], families of continuum cardinality of pairwise non isomorphic Whitney spaces were presented. The spaces differed in diametral dimension. The article [5] was devoted to the following generalization of the above set $K^{(\alpha)}$. Now we fix not only a sequence $\mathcal{L} = (\ell_s)_{s=1}^\infty$ of lengths of basic intervals, but also a sequence of integers $\mathcal{N} = (N_s)_{s=1}^\infty$ with $N_s \ge 2$ for all s. Then $K(\mathcal{L}, \mathcal{N}) = \bigcap_{s=0}^\infty E_s$, where $E_0 = I_{0,1} = [0,1]$ and E_s, $s \ge 1$, is a union of $N_1 N_2 \cdots N_s$ disjoint closed intervals $I_{s,k}$ of length ℓ_s and E_{s+1} is obtained by replacing each interval by N_{s+1} disjoint subintervals $I_{s+1,j}$ of length ℓ_{s+1} with $N_{s+1}-1$ gaps of equal lengths. As for the set $K^{(\alpha)}$, if $s \geq 2$ then let α_s satisfy $\ell_s = \ell_{s-1}^{\alpha_s}$. If in particular $\alpha_s = \alpha$ and $N_s = N$ for all s then we denote the set $K(\mathcal{L}, \mathcal{N})$ by K_N^α, so K_2^α is K^α in our previous notation. Also for the above family of sets K_N means that only N_s is constant.

For $0 < \lambda < \infty$ and the function h_0 from (6.5) we consider the Hausdorff h_0^λ-measure of a compact set K. As in the definition of Hausdorff dimension, there exists a critical value $\lambda_0 = \lambda_0(K)$, $0 \leq \lambda_0 \leq \infty$, such that $\mu_{h_0^\lambda}(K) = \infty$ for $\lambda < \lambda_0$ and $\mu_{h_0^\lambda}(K) = 0$ for $\lambda > \lambda_0$. Since the function h_0 corresponds to the logarithmic measure we will say that the value λ_0 is the *logarithmic dimension* of K.

Suppose that the limit $\lambda_0 = \lim_s \frac{\log N_s}{\log \alpha_s}$ exists in the set of extended real numbers. Then, by [5], Prop. 1, $\lambda_0(K((\ell_s),(N_s))) = \lambda_0$. Later it was shown in [55] that the true value of the logarithmic dimension of this set is the corresponding $\liminf$ of the above fraction.

The value $\lambda_0 = \lambda_0(K_N)$ can be applied for the characterization of the extension property of such sets. Also, the logarithmic dimension of sets is involved in the description of diametrical dimension for spaces: by Corollary 2 in [5], 4,

$$\Gamma(\mathcal{E}(K_N^\alpha)) = \{(\gamma_n) : \exists M : \ \gamma_n \exp(-Mn^{\frac{1}{\lambda_0}}) \to 0 \text{ as } n \to \infty\}.$$

An easy computation shows that in the family of sets $(K^\alpha)_{\alpha>1}$ the spaces $\mathcal{E}(K^\alpha)$ corresponding different values of the parameter are not isomorphic.

Another example of using the diametral dimension for $\mathcal{E}(K)$ is [43], where Whitney jets were considered on the set $K = \{0\} \bigcup \bigcup_{k=0}^\infty \{a_k\}$, as K_φ above, but without additional regularity conditions as in [57]. In the case of *sparse* sequences, which means that for some constant $Q \geq 1$ the inequality $a_k - a_{k+1} \geq a_k^Q$ is valid for all k, the diametral dimension distinguishes spaces. The family of sequences $(\exp(-\log^\alpha k)_{k=1}^\infty)_{\alpha \geq 1}$ gives pairwise not isomorphic spaces of Whitney jets $\mathcal{E}(K)$. But for *thick* sequences, now for each Q the inequality $a_k - a_{k+1} < a_k^Q$ asymptotically valid, the diametral dimension of $\mathcal{E}(K)$ is minimal possible, that is $\Gamma(\mathcal{E}(K)) = \Gamma(s)$.

As for the restriction spaces $C_\infty(K_\varphi), \varphi \in \Phi$, from Section 11, thanks to the knowledge of bases, their complete isomorphic classification was presented in section 13 of [57] using the dual diametral dimension. It is given as follows

$$\Gamma'(\mathcal{X}) = \{\gamma = (\gamma_n)_{n=0}^\infty : \exists p \forall q : \gamma_n / d_n\,(U_q, U_p) \to 0 \text{ as } n \to \infty\}\,. \tag{12.2}$$

Suppose $\varphi_1, \varphi_2 \in \Phi$ are given. If both are from Φ_{sub} then $C_\infty(K_{\varphi_1}) \simeq C_\infty(K_{\varphi_2})$ if and only if there is a positive constant C so that

$$C^{-1}\,\varphi_1(k) \leq \varphi_2(k) \leq C\,\varphi_1(k) \ \text{ for all } \ k.$$

If $\varphi_1, \varphi_2 \in \Phi \setminus \Phi_{\text{sub}}$ then $C_\infty(K_{\varphi_1}) \simeq C_\infty(K_{\varphi_2})$ if and only if for each p there exists m and k_0 so that

$$\varphi_2(k-p-m) \leq \varphi_1(k-p) \leq \varphi_2(k-p+m) \ \text{ for } \ k \geq k_0.$$

Of course, $C_\infty(K_{\varphi_1}) \not\simeq C_\infty(K_{\varphi_2})$ for index functions of different characters of growth rate.

It should be noted that the diametral dimension can be used to prove the existence of a basis in some Whitney spaces. By [6], if a complemented subspace $\mathcal{X}$ of s has a stable diametral dimension ($\Gamma(\mathcal{X} \times \mathcal{X}) = \Gamma(\mathcal{X})$) then $\mathcal{X}$ has a basis. This method was used in [64] for K_N^α and in [115] for the classical Cantor ternary set and for the Sierpinski triangle.

13 Interpolating Invariants and Vogt's Splitting Theory

The spaces $A(\mathbb{D}), A(\mathbb{C})$ are not isomorphic since they have different diametral dimensions.

There is another reason for the lack of such isomorphism. Dragilev introduced in [24] two disjoint invariant classes of Fréchet spaces:

$\mathcal{X} \in d_1$ if $\exists p\, \forall q\, \exists r, C > 0 : |x|_q^2 \le C\, |x|_p |x|_r$ for $x \in \mathcal{X}$,

$\mathcal{X} \in d_2$ if $\forall p\, \exists q\, \forall r \exists C > 0 : |x|_p |x|_r \le C\, |x|_q^2$ for $x \in \mathcal{X}$.

Since $A(\mathbb{C}) \in d_1, A(\mathbb{D}) \in d_2$, they are not isomorphic.

The condition for d_1 is a kind of logarithmic convexity of the sequence $(|x|_j)_{j=1}^\infty$. The same is valid for d_2 but with respect to the dual norms: d_2 is equivalent to

$$\forall p\, \exists q\, \forall r \exists C > 0 : (|x'|_{-q})^2 \le C\, |x'|_{-p} |x'|_{-r} \quad \text{for} \quad x' \in \mathcal{X}'. \tag{13.1}$$

There are numerous generalizations and even different notations of these invariants: d_1 is (DN), $d_2 = (\overline{\Omega})$. An important feature of interpolation invariants is that they can be used not only to distinguish non isomorphic spaces, but also to obtain more general results about the structure of spaces and properties of operators between them. For example, the compactness of every continuous mapping from the space of power series of finite type to the space of power series of infinite type was shown in [123].

The most significant application of interpolation invariants is a comprehensive splitting theory for a short exact sequence

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{\pi} E \longrightarrow 0 \tag{13.2}$$

of Fréchet-Hilbert spaces. First, for the case $G = s$ with NF spaces F, E it was proved in [110] and [111] that the above sequence splits if and only if the space E has a dominating norm and F has the property (Ω). The last is the following weakened version of (13.1):

$$\forall p\, \exists q\, \forall r \forall \theta \in (0,1)\, \exists C > 0 : |x'|_{-q} \le C\, |x'|_{-p}^{1-\theta} |x'|_{-r}^{\theta} \quad \text{for} \quad x' \in \mathcal{X}'.$$

Tidten applied this characterization to (4.7). He proved in [100] that the ideal (4.8) has (Ω) for each compact set K, so (4.7) splits if and only if $\mathcal{E}(K)$ has DN. Later in [31], Th.2.1 it was shown that every closed ideal of $C^\infty(\mathbb{R})$ satisfies (Ω), so we can use the above characterization also for the sequence (11.1).

The same conditions for spaces F, E as above characterize the splitting of (13.2) when G is any power series space of infinite type. However, for a general Frechet-Hilbert space G, possibly even a nuclear one, the (DN) property of E and the (Ω) property of F are only sufficient for the splitting of (13.2), see [78], 30. For this reason we cannot state that (11.2) does not split if $\varphi \notin \Phi_{\text{sub}}$.

Generalizations of the property (DN) have proven useful for isomorphic classifications of Whitney spaces. Independently, the similar property (DN_Φ) in [102] and the class (D_ϕ) in [32] were introduced to present families of continuum cardinality of pairwise non-isomorphic spaces $\mathcal{E}(K)$ and $C^\infty(\overline{\Omega})$ for compact sets K and $\overline{\Omega}$ of sharp cusp type, so $C^\infty(\overline{\Omega}) \simeq \mathcal{E}(\overline{\Omega})$. We follow here [32] since the corresponding family is more simple and the invariant (D_ϕ) was successfully used for other spaces as well.

In this section, let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function with $\varphi(t) \ge t$. A Fréchet space $\mathcal{X}$ with a fundamental increasing system of semi-norms $(\|\cdot\|_p)_{p=0}^\infty$ belongs to the class (D_φ) if $\exists p \in \mathbb{Z}_+\ \forall q \in \mathbb{N}\ \exists r \in \mathbb{N}, m > 0, C > 0$ such that

$$\|x\|_q \le \varphi^m(t) \|x\|_p + \frac{C}{t} \|x\|_r, \quad \text{for} \quad t > 0,\, x \in \mathcal{X}.$$

In the case $\varphi(t) = t$ the invariant (D_φ) coincides with $(\underline{DN})$, see [78], p. 368 and Prop. 29.18 for the definition of $(\underline{DN})$ and its importance in the structure theory of power series spaces of finite type. The strong form of the condition when $m = 1$ coincides with the property (DN_φ).

Let ψ be a positive increasing on $(0,1)$ function with $\psi(x) \le x$ and the domain Ω_ψ be defined as in (3.3). As in Section 6, we assume that the boundary of Ω_ψ is Lipschitz at the point $(1, \psi(1))$. Then, by Theorem 3.1 in [33], see also [32], the space $C^\infty(\overline{\Omega}_\psi)$ has the property (D_φ) if and only if there are constants M, N and C such that the inequality

$$\psi^{-1}(x) \le C\,\varphi^M(x^{-N})$$

is valid asymptotically for $x \to 0$. This allows us to present a simple family of functions $(\psi_\lambda)_{\lambda \ge 1}$ so that the spaces $\mathcal{X}_\lambda = C^\infty(\overline{\Omega}_{\psi_\lambda})$ are not isomorphic for different index values.

Example 13.1. *Let $\psi_\lambda(x) = \exp(-\log^\lambda(1/x))$ if $0 < x < e^{-1}$ and $\psi_\lambda(x) = x$ if $e^{-1} \le x < 1$. Then $\mathcal{X}_\lambda \not\simeq \mathcal{X}_\mu$ for $\lambda \ne \mu$.*

The same invariant (D_φ) was used in [39], where the Cantor-type sets $K^{(\alpha)}$ associated with the sequence (6.1) was considered when $\lim_{s\to\infty} \alpha_s = \infty$. By Theorem 2 in [39], the space $\mathcal{E}(K^{(\alpha)})$ has the property (D_φ) if and only if

$$\forall k\ \exists M: \quad \ell_s \ge \varphi^{-M}(\ell_{s-k}^{-1}) \ \text{ for } \ s > k.$$

Example 13.2. *Let $\ell_1 = 1/e$, $(\alpha)_\lambda = (\exp s^\lambda)_{s=2}^\infty$ for $\lambda > 1$ and $\mathcal{X}_\lambda = \mathcal{E}(K_\lambda^{(\alpha)})$. Suppose $\lambda < \mu$. Take ρ with $\lambda/(\lambda+1) < \rho < \mu/(\mu+1)$. Given ρ, let $\varphi(t) = t$ for $0 \le t \le e$ and $\varphi(t) = t^{\exp(\ln\ln t)^\rho}$ for $t > e$. Then the space $\mathcal{X}_\lambda$ has the property (D_φ), while $\mathcal{X}_\mu$ does not.*

Also, the invariant (D_φ) distinguishes the spaces of Whitney functions on the sets $K(\mathcal{L}, \mathcal{N})$ from the previous section with $N_s \to \infty$. With the natural restrictions $\ell_s < \hbar_s$ and $\hbar_s \ge \ell_{s-1}^Q$ for some $Q \ge 1, n \ge 2$, by Theorem 3 in [39], the space $\mathcal{E}(K(\mathcal{L}, \mathcal{N}))$ has the property (D_φ) if and only if

$$\exists M: \quad \ell_s \ge \varphi^{-M}(\ell_{s-1}^{-M}) \ \text{ for } \ s > 1.$$

Example 13.3. *Let ℓ_1 and $(\alpha)_\lambda$ be the same as in the previous example and $\mathcal{L}_\lambda$ be the sequence of lengths determined by the given λ. Now let $\hbar_s = \ell_{s-1}^2$. Fix natural numbers N_s with $N_s > \ell_{s-1}/(\ell_s + \hbar_s)$. Then $N_s \to \infty$ as $s \to \infty$, so $K(\mathcal{L}_\lambda, \mathcal{N})$ satisfies the desired condition. Let $\mathcal{X}_\lambda = \mathcal{E}(K(\mathcal{L}_\lambda, \mathcal{N}))$. We fix λ, μ, ρ and $\gamma(t)$ as in the previous example. Then $\mathcal{X}_\lambda$ has the property (D_φ), while $\mathcal{X}_\mu$ does not.*

And that's not all. There are also compound invariants.

14 Zahariuta's Compound Invariants

NF spaces $\mathcal{X} = \bigcap_p \mathcal{H}_p$ is a projective limit of a sequence of Hilbert spaces with a compact embedding $J_{pq} : \mathcal{H}_q \hookrightarrow \mathcal{H}_p$. Its singular numbers (the eigenvalues of self-adjoint operator $J_{pq}^* J_{pq}$) form a null sequence (σ_k) of non-negative numbers.

Mityagin proposed in [80], Prop. 16, to consider the asymptotic behavior of the numbers σ_k for various spaces in a given interval $I(t) = [a(t), b(t)]$ with $0 < a(t) < b(t)$ as $t \to \infty$. If the spaces are non-isomorphic, then the corresponding asymptotics must be different.

A nice geometric interpretation of this idea was suggested in [122] by Zakhariuta: calculate the diametral dimension for *synthetic* neighborhoods. Given $U_r \subset U_q \subset U_p$ (with compact embeddings $(\mathcal{X}, |\cdot|_r) \hookrightarrow (\mathcal{X}, |\cdot|_q)$ and $(\mathcal{X}, |\cdot|_q) \hookrightarrow (\mathcal{X}, |\cdot|_p)$), large t and small $\tau > 0$, let $\tilde{V} := \tau U_p \bigcap t U_r$ and

$\tilde{U} := \operatorname{conv}(\tau U_p \bigcup t U_r)$. Then $d_n(\tilde{V}, U_q)$, $d_n(U_q, \tilde{U})$ and $d_n(\tilde{V}, \tilde{U})$ go to zero for $n \to \infty$. The following counting characteristic is useful for analyzing how quickly the zero sequence converges. Let us consider it first for the standard Kolmogorov diameter $d_n(U_q, U_p)$:

$$m_0(t) = m_0(t, U_p, U_q) := \min_{L \in \mathcal{L}} \{\dim L : tU_q \subset U_p + L\} \quad \text{for } t > 0,$$

where $\mathcal{L}$ is the collection of all finite-dimensional subspaces of $\mathcal{X}$. It is easy to check that $m_0(t) = \left|\{n : d_n(U_q, U_p) > \frac{1}{t}\}\right|$, where $|A|$ denotes the cardinality of a finite set A.

If $\mathcal{X}$ is a Schwartz space, then $m_0(t, U_p, U_q)$ takes finite values for sufficiently apart p and q. It is straightforward to show that

$$(\gamma_n) \in \Gamma(E) \Longleftrightarrow \forall p\, \exists q : \forall C\, \exists n_0 : m_0(C\gamma_n, U_p, U_q) \le n \ \text{ for } n \ge n_0.$$

If Fréchet spaces $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic, then

$$\forall p_1\, \exists p\, \forall q\, \exists q_1, C : m_0^{(\mathcal{Y})}(t, V_{p_1}, V_{q_1}) \le m_0^{(\mathcal{X})}(Ct, U_p, U_q), \quad \text{for } t > 0$$

and vice-versa. Here $(U_p)_{p=1}^{\infty}, (V_p)_{p=1}^{\infty}$ are bases of neighborhoods of zero for spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively.

In an exactly similar way, for sufficiently apart $p, q, r \in \mathbb{N}$ with $p < q < r$ and $0 < \tau < t$ the counting function corresponding to $d_n(\tilde{V}, U_q)$

$$m(\tau, t) = m(p, q, r, \tau, t) := \min_{L \in \mathcal{L}} \{\dim L : \tau \cdot U_p \bigcap t \cdot U_r \subset U_q + L\} \tag{14.1}$$

is finite. This function is an isomorphic characteristic of (F)-spaces in the following sense.

If F-spaces $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic, then

$$\forall p \exists p_1 \forall q_1 \exists q \forall r \exists r_1, C > 0 : \ m^{(\mathcal{Y})}(p_1, q_1, r_1, \tau, t) \le m^{(\mathcal{X})}(p, q, r, C\tau, Ct), \ t > 0, \tau > 0$$

and analogous condition holds after interchange of $m^{(\mathcal{X})}$ and $m^{(\mathcal{Y})}$.

In the same manner we can write the counting function corresponding to $d_n(U_q, \tilde{U}), d_n(\tilde{V}, \tilde{U})$ and other synthetic neighborhoods. The number of components in this type neighborhood is not limited, This leads to the appearance of numerous LTI with unlimited complexity. Obviously, calculations for increasingly complex compound invariants become increasingly technically challenging.

These invariants generalize the invariants of both types considered above. If we take large enough $\tau = \tau(t)$, then $m(\tau, t) = |\{n : d_n(U_r, U_q) > \frac{1}{t}\}|$ is the counting function corresponding to the diametral dimension.

On the other hand, one can describe known interpolation invariants in the terms of asymptotic degeneration of the function m. For example, an F-space $\mathcal{X}$ has the property D_φ if and only if $\exists p : \forall q\, \exists r$ (all natural numbers) and positive M, ε, t_0 such that

$$m(\varphi^{-M}(t), \varepsilon t) = 0, \ t > t_0.$$

The first application of compound invariants was in [125], where Zaharjuta presented a continuum family of mutually non-isomorphic spaces of the form $A(D)$, where D is a pseudo-convex Reinhardt domain in $\mathbb{C}^N$. Elementary large families of pairwise non-isomorphic NF-spaces were known earlier. For example, let $\beta_\lambda = (k^\lambda)_{k=1}^{\infty}$ for $\lambda > 1$. Then the spaces $E_\infty(\beta_\lambda)$ are not isomirphic for different indices. This can be shown, for example, using the dual diametral dimension (12.2) since in the nuclear case $\Gamma'(E_\infty(\beta)) = E_\infty(\beta)$.

In turn, the corresponding family from [125] is quite nontrivial. Before this, the maximum result in this direction was [86], where four classes of mutually non-isomorphic spaces $A(D)$ were found.

Composite invariants have proven to be very effective in classifying Whitney spaces. In a number of cases, families of compact sets have been presented for which the corresponding Whitney spaces cannot be distinguished using interpolation invariants, but nevertheless they are pairwise non-isomorphic. The following example is taken from [37].

Example 14.1. *Let for $\lambda > 1$ the space $\mathcal{X}_\lambda$ be $\mathcal{E}(\{0\} \bigcup \bigcup_{k=3}^{\infty} [a_k^{(\lambda)}, b_k^{(\lambda)}])$ with $b_k^{(\lambda)} = \exp(-(\ln k)^\lambda)$, $a_k^{(\lambda)} = b_k^{(\lambda)} - \exp(-\exp(\ln k)^\lambda)$, so the length of k-th interval is $\psi_0(b_k^{(\lambda)})$ for the function ψ_0 from (3.3). Assume that $\lambda \neq \mu$ and $\mathcal{X}_\lambda$ has the property D_φ for some φ. Then $\mathcal{X}_\mu$ also has D_φ with the same φ. However, $\mathcal{X}_\lambda \not\simeq \mathcal{X}_\mu$ as they have different asymptotic of the function (14.1).*

A much more sophisticated situation was considered in [4]. At that time, a basis in the space $\mathcal{E}_0(K)$ was known for the set $K = \{0\} \bigcup \bigcup_{k=1}^{\infty} [a_k, b_k]$, see Section 8.

Let now $\ell_k = b_k - a_k \searrow 0, \hbar_k = a_k - b_{k+1} \searrow 0$ as $k \to \infty$ and $\hbar_k \geq b_k^Q$ for some constant Q and all k. Then $\mathcal{E}_0(K)$ is isomorphic to a Köthe space with the matrix $A = (a_{nkp})_{n,k=1,p=0}^{\infty}$, where

$$a_{nkp} = n^p \ell_k^{-u_p} \hbar_k^{-p}, \;\; u_p = \min(n, p).$$

As written above, the counting function m from (14.1) corresponds to $d_n(\tau\, U_p \bigcap t\, U_r, U_q)$. In a similar way, let m_1 correspond to $d_n(U_p^\varepsilon\, U_r^{1-\varepsilon} \bigcap t\, U_r, U_q)$, where ε is chosen so that $q < \varepsilon\, p + (1-\varepsilon)\, r$ and m_2 corresponds to $d_n(U_q \bigcap U_p^\varepsilon\, U_r^{1-\varepsilon}, \overline{conv}(\tau U_p \bigcup U_q))$.

Thus we have four linear topological invariants of increasing complexity: D_φ, m, m_1, m_2. Using the above Köthe spaces, for each invariant we can present a continuum of mutually non-isomorphic $\mathcal{K}(A)$ non-distinguishable by previous invariants. It shows that the topological structure of Whitney spaces is quite complex even for sets of the given type.

Obviously, using linear topological invariants one can only prove the absence of an isomorphism between given spaces $\mathcal{X}$ and $\mathcal{Y}$. Positive results in this direction can be obtained by directly representing the isomorphism. Knowing the bases in $\mathcal{X}$ and $\mathcal{Y}$ simplifies the problem, since isomorphisms between spaces with bases are usually diagonal or quasi-diagonal.

The most applications of the compound invariants were devoted to the quasi-equivalence problem.

15 Problem of Quasi-Equivalence of Bases

Suppose a Banach space $\mathcal{X}$ has an unconditional basis $(e_k)_{k=1}^{\infty}$. This means that for every $x \in \mathcal{X}$ the series (1.1) converges unconditionally. Then any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence of nonzero scalars $(\rho_k)_{k=1}^{\infty}$ generate another basis $(f_k)_{k=1}^{\infty}$ with $f_k = \rho_k\, e_{\sigma(k)}$. If any unconditional basis $(f_k)_{k=1}^{\infty}$ of $\mathcal{X}$ can be represented in this form, we say that $\mathcal{X}$ has unique unconditional basis (up to permutation). The uniqueness property for a basis is quite exceptional in the context of Banach spaces: by [74], only c_0, ℓ_1, ℓ_2 and some more sophisticated spaces as $c_0 \times \ell_2, \ell_1 \times \ell_2, c_0(\ell_1), \ldots$ have unique (up to permutation) unconditional basis.

The opposite situation is observed in the class of NF-spaces, where all bases are absolute. Similar in the above sense bases are called quasi-equivalent. Namely, two bases $(e_k)_{k=1}^{\infty}$ and $(f_k)_{k=1}^{\infty}$ in an NF-space $\mathcal{X}$ are *quasi-equivalent* if there exist a permutation σ, scalars $\rho_k \neq 0$, and an isomorphism $T : \mathcal{X} \to \mathcal{X}$ such that $Te_k = \rho_k\, f_{\sigma(k)}$ for each k.

This section gives only a brief overview of results for NF-spaces. Similar to the absoluteness of the basis and interpolation invariants, the pioneering result here is due to Dragilev [23]: all bases in $A(\mathbb{D})$ are quasi-equivalent. This result was generalized by Mityagin in [79] and [80] to the case of Hilbert scales. There, a combinatorial technique method was proposed, which was subsequently widely used in this area: by means of the Hall-König theorem to compare Köthe matrices for different bases.

All bases are quasi-equivalent in $\mathcal{X}$ with a regular basis. It was shown independently in [19] and [69] and generalized to bases of type G_1 or G_∞ in [2].

In a number of works, Zaharyuta himself and with his co-authors showed the quasi-equivalence of bases in many classes of spaces. Special Köthe spaces, *first type power power Köthe spaces* with $c_k > 0, 0 < \lambda_k < 1$ for $k \in \mathbb{N}$

$$E(\lambda, c) = \mathcal{K}[\exp(-1/p + \lambda_k\, p)\, c_k]$$

were considered in [123], [17] and some other papers. The interest in these spaces is due to the fact that they represent (up to isomorphism) Cartesian and tensor products of power series spaces of different type, that is $E_0(\alpha) \times E_\infty(\beta)$ and $E_0(\alpha)\hat{\otimes}_\pi E_\infty(\beta)$. The last space is related with the spaces $C^\infty_{\mathcal{X}}$ of infinitely differentiable functions $f : [0,1] \longrightarrow \mathcal{X}$ for a locally convex space $\mathcal{X}$. By [58], $C^\infty_{\mathcal{X}} \simeq C^\infty[0,1]\hat{\otimes}_\pi \mathcal{X}$.

Tensor products of power series spaces with their duals $E_\infty(\alpha)\hat{\otimes}_\pi E'_\infty(\beta)$ were analyzed in [35]. Now the products $E_\infty(\alpha)\hat{\otimes}_\pi E'_\infty(\beta)$ can be identified with the space of linear continuous operators. For example,

$$C^\infty[0,1]\hat{\otimes}_\pi E'_\infty(\beta) \simeq L(C^\infty[0,1], E_\infty(\beta)).$$

Here the dual space $E'_\infty(\beta)$, which is the inductive limit of $l_\infty(\exp(-p\,\beta_k))$ is represented as a projective Köthe type space, but with uncountable indices.

In all cases, isomorphic classification was carried out using different types of equivalence of exponent sequences or special characteristics of their lacunarity. More complicated LTI than those described in the previous section give the method of $m-$rectangular characteristics to analyze $E(\lambda, c)$. For each $m \in \mathbb{N}$, we fix m rectangles in the region $\mathbb{R}_+ \times (0,1]$ and count the number of points (c_k, λ_k) in the union of these rectangles.

What can be said about quasi-equivalence of bases considered in Sections 7–11? There are very special bases in the space $C^\infty[-1,1]$, but, by Theorem 1 in [24], if a NF-space has a regular basis then any another its basis became regular after a suitable permutation. Different choices of the function l in Section 7 and the sequence (k_s) in Section 11 are possible, but they give very similar bases. More promising in this regard is the space $\mathcal{E}(K^{(\alpha)})$ from Section 10. If $\alpha_s \geq 2$ for all s then the space has, on the one hand, a strict polynomial basis, and on the other hand, a multitude of different local bases with arbitrarily small supports of the basis elements. Of course, the bases in $\mathcal{E}(K^{(\alpha)})$ are not regular. For that reason, the question of quasi-equivalence of these bases was raised in [44], p.359. An affirmative answer was given in [55] by explicitly constructing the desired isomorphism.

The conjecture on the quasi-equivalence of bases in NF-spaces was implicitly stated in [80] (Problem 12) and in [81], see also Problem 8 in [7]. We can quote Zobin ([129]), who wrote twenty five years ago: “There was a common belief at that time that the conjecture will be proven very soon. To everybody’s great surprise, it is still an open question.” Two decades later, the status of the hypothesis remains the same.

16 Some Open Problems

In conclusion, we collect here the open problems mentioned above. The number of the corresponding section is indicated in parentheses.

1. (Section 5) Is Pleśniak’s approach universal? Is the operator (5.2) bounded for any compact set K with the extension property? If so, can its boundedness be proven by comparing the Markov

$M_n^{(p)}(K)$-factors (see the definition in Section 4 of [56]) and the best approximations $E_n(f, K)$ to f on K, which can be estimated via $||f||_q$ with $q = q(p)$. It can be assumed that the "worst" mixed compact set $K = [-1, 0] \cup K^2$ with the extension property provides a counterexample.

More generally, the phenomenon of ultra-fast convergence $E_n(f, K) \to 0$ as $n \to \infty$, where $f \in \mathcal{E}(K)$ for a small set K, requires a mathematical description. It is interesting to understand at least in terms of what functions, depending on K, Jackson-type inequalities should be written.

2. (Section 5) Is the approach by Frerick, Jordá, and Wengenroth universal? Is it possible to choose local measures $\mu_{\alpha,j}$ such that the operator (5.4) is bounded for any compact set K with the extension property?

3. (Section 5) Are the extensions obtained by the three methods mentioned above different? Is there a unifying approach that combines them?

4. (Section 6) Is it possible to characterize the extension property (even for polar sets) in terms of logarithmic potential theory? Condition (6.4) assumes the existence of a sequence of compact sets with $K_j \searrow K$ so that for each $x \in K$ and $\varepsilon > 0$ with the set $A_j := K_j \bigcap [x - \varepsilon, x + \varepsilon]$ the Robin constants $Rob(A_j)$ have uniformly regular growth (perhaps to ∞) as $j \to \infty$.

5. (Section 6) Find the boundaries of uncertainty zones both for the growth of Markov factors (for general compact sets) and for dimension functions (for dimensional Cantor sets).

6. (Section 6) Is the condition at the end of Section 6 a geometric characterization of the extension property for $K \subset \mathbb{R}$?

7. (Section 8) Is it possible to find a measure μ on the domain Ω_{ψ_0} from (3.3) such that the polynomials orthogonal with respect to μ form a basis in the space $\mathcal{E}(\bar{\Omega}_{\psi_0})$? If such a measure exists, then its density with regard to the Lebesgue measure will increase rapidly near the origin.

   A similar question for running duck sets is also interesting and perhaps simpler.

8. (Section 10) Prove or disprove the hypothesis at the end of the section.

9. (Section 11) By Theorem 8.1 in [54], the system $(f_{nk})_{n,k=0}^{\infty}$ is a basis in $\mathcal{E}(K_\varphi)$ for $\varphi \in \Phi_{\text{sub}}$. It is not difficult to verify that in the case of a function φ of exponential growth it does not have the basis property. The author assumes that the space $\mathcal{E}(K_\varphi)$ with $\varphi \in \Phi \setminus \Phi_{\text{sub}}$ does not have a basis at all.

10. (Section 15) Decade after decade, the problem of quasi-equivalence of bases in nuclear Fréchet spaces cannot be solved.

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