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Survey Article

Positive Linear Operators and Entropies

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Abstract: The study of positive linear operators is closely linked to the study of probability distributions. In this paper, we investigate the relationship between classical positive linear operators, commonly used in approximation theory, and the entropies associated with probability distributions. These entropies can be expressed in terms of special functions, particularly Heun functions, convex functions, and logarithmically convex functions, which satisfy specific equalities and inequalities, and for which various bounds can be established. These properties are important for a wide range of applications. This is a non-exhaustive survey: we present results without proofs and provide detailed bibliographic references.

Keywords: Positive linear operators; probability distributions; information potential; entropies; Heun functions; convex functions.

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1 Introduction

Let $I \subset \mathbb{R}$ be an interval and let $p_k(x)$, $k = 0, 1, \dots$ continuous functions defined on I such that $\sum_{k=0}^{\infty} p_k(x) = 1$, $x \in I$. Then $p(x) := (p_k(x))_{k\geq 0}$ can be viewed as a parameterized probability distribution, and equivalently, one can use $p(x)$ to construct a discrete positive linear operator of the form

$$Lf(x) = \sum_{k=0}^{\infty} f(x_k)p_k(x), \quad x \in I,$$

where $x_k \in I$, $k \geq 0$, and f is in a suitable space of functions defined on I.

As concrete examples we may think of the binomial, Poisson and negative binomial distributions, associated respectively with the Bernstein, Szász-Mirakjan and classical Baskakov operators.

In this paper the function $S(x) := \sum_{k=0}^{\infty} p_k^2(x)$, $x \in I$, will play a prominent role. It is the information potential, see [29], also called the index of coincidence. In probabilistic terms, $-\log S(x)$ is the Rényi entropy of order 2 (see [31], [36], [44]) and $1 - S(x)$ is the Tsallis entropy of order 2 (see [31], [41], [42]). On the other hand $S(x)$ is involved in the study of the "degree of non-multiplicativity" of the operator L using the Chebyshev-Grüss type inequalities. For results of this type see [2], [4], [16], [17], [40].

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Example 1.1. (see [17, Th. 9]) *Let L be defined on the space $B(I)$ of all real-valued, bounded functions on I. Then*

$$|L(fg)(x) - Lf(x)Lg(x)| \leq \frac{1}{2}(1 - S(x))osc(f)osc(g),\ f, g \in B(I),\ x \in I,$$

where $osc(f) := sup\{|f(x_j) - f(x_i)| : i, j \geq 0\}$.

We will see that $S(x)$ is a Heun function. The Heun equation has the form

$$y''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right) y'(x) + \frac{\alpha\beta x - q}{x(x-1)(x-a)} y(x) = 0,$$

where $a \neq 0, 1$ and $\alpha + \beta + 1 = \gamma + \delta + \epsilon$.

A solution normalized by $y(0) = 1$ is called a local Heun function, denoted by $Hl(a, q; \alpha, \beta; \gamma, \delta; x)$. For details see [12], [24], [25], [26], [37]. Applications of Heun functions in Physics are presented in [22].

Let $c \in \mathbb{R}$. Define $I_c := \left[0, -\frac{1}{c}\right]$ if $c < 0$, and $I_c := [0, +\infty)$ if $c \geq 0$. Let $n > 0$ be a real number, $k \in \mathbb{N}_0$ and $x \in I_c$. Denote

$$p_{n,k}^{[c]}(x) := (-1)^k \binom{-\frac{n}{c}}{k} (cx)^k (1 + cx)^{-\frac{n}{c} - k}, \quad \text{if } c \neq 0,$$

$$p_{n,k}^{[0]}(x) := \lim_{c \to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!} e^{-nx}, \quad \text{if } c = 0.$$

The functions $p_{n,k}^{[c]}$ are used in the construction of the following sequence of positive linear operators:

$$L_n^{[c]} f(x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}^{[c]}(x), \quad x \in I_c.$$

They were introduced in [8]. For details, see [9], [18], [43].

It is not difficult to verify that $\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1$. In what follows we shall suppose that $n > c$ if $c \geq 0$, or $n = -cl$ with some $l \in \mathbb{N}$ if $c < 0$.

For $c = -1,\ 0,\ 1$, the operators $L_n^{[c]}$ reduce to the Bernstein, Szász-Mirakjan, and Baskakov operators, respectively.

In the context delineated above, we will be interested in properties of the information potential and the associated entropies viewed as Heun functions, convex and logarithmically convex functions which satisfy specific equalities/inequalities and for which bounds can be established. These properties are important for applications.

The structure of the paper is as outlined below. Section 2 is concerned with the family of probability distributions $(p_{n,k}^{[c]}(x))_{k\geq 0}$. The distributions corresponding to $c = -1$, respectively $c = 1$, are presented in Sections 3 and 4. The probability distributions associated with Bleimann-Butzer-Hahn, respectively Meyer-König and Zeller operators are considered in Sections 5 and 6. The distribution $(p_{n,k}^{[0]}(x))_{k\geq 0}$ corresponding to the Szász-Mirakjan operators is the object of study in Section 7. Conclusions and further work are presented in Section 8.

We present a non-exhaustive survey, focusing on key results without proofs, but supported by thorough bibliographic references.

2 Main Results

To the probability distribution $(p^{[c]}_{n,k}(x))_{k\geq 0}$ we associate the function

$$S_{n,c}(x) := \sum_{k=0}^{\infty} \left(p^{[c]}_{n,k}(x)\right)^2, \quad x \in I_c.$$

The following two theorems were obtained in [32] using results from [9].

Theorem 2.1. (see [32]) *Let $c \neq 0$. Then*

$$S_{n,c}(x) = (1+cx)^{-\frac{2n}{c}} \sum_{k=0}^{\infty} \left(\frac{n(n+c)\dots(n+(k-1)c)}{k!}\right)^2 \left(\frac{x}{1+cx}\right)^{2k},$$

$$S_{n,c}(x) = (1+cx)^{-\frac{2n}{c}} {}_2F_1\left(\frac{n}{c}, \frac{n}{c}; 1; \left(\frac{cx}{1+cx}\right)^2\right),$$

$$S_{n,c}(x) = \frac{1}{\pi}\int_0^1 \left(t+(1-t)(1+2cx)^2\right)^{-\frac{n}{c}} \frac{dt}{\sqrt{t(1-t)}}. \tag{2.1}$$

If $c = 0$, then

$$S_{n,0}(x) = e^{-2nx} \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(k!)^2},$$

$$S_{n,0}(x) = e^{-2nx} I_0(2nx),$$

$$S_{n,0}(x) = \frac{1}{\pi}\int_{-1}^1 e^{-2nx(1+t)} \frac{dt}{\sqrt{1-t^2}}.$$

Here ${}_2F_1$ is the Gauss hypergeometric function and I_0 is the modified Bessel function of first kind of order 0.

Theorem 2.2. (see [32])

(i) *The function $S_{n,c}$ is a solution to the differential equation*

$$x(1+cx)(1+2cx)y''(x) + (4(n+c)x(1+cx)+1)y'(x) + 2n(1+2cx)y(x) = 0, \quad x \in I_c.$$

(ii) *If $c \neq 0$, the function $H_{n,c}(x) := S_{n,c}\left(-\frac{x}{c}\right)$ is a solution to the Heun equation*

$$y''(x) + \left(\frac{1}{x} + \frac{1}{x-1} + \frac{\frac{2n}{c}}{x-\frac{1}{2}}\right)y'(x) + \frac{\frac{2n}{c}x - \frac{n}{c}}{x(x-1)\left(x-\frac{1}{2}\right)}y(x) = 0,$$

i.e.,

$$H_{n,c}(x) = Hl\left(\frac{1}{2}, \frac{n}{c}; \frac{2n}{c}, 1; 1, 1; x\right).$$

The following conjecture was formulated in [30].

Conjecture 2.1. (see [30]) *For $c \in \mathbb{R}$, $S_{n,c}$ is logarithmically convex, i.e., $logS_{n,c}$ is convex.*

In [1] U. Abel, W. Gawronski and Th. Neuschel obtained a stronger result for $c \geq 0$.

Theorem 2.3. (see [1]) *For $c \geq 0$, the function $S_{n,c}$ is completely monotone in the sense of Bernstein's theorem, i.e.,*

$$(-1)^m \left(\frac{d}{dx}\right)^m S_{n,c}(x) > 0, \quad x \geq 0, m \geq 0.$$

Consequently, for $c \geq 0$, $S_{n,c}$ is logarithmically convex, and hence convex.

For the proof of the conjecture when $c < 0$ see the next section.

The following three results can be found in [3].

Theorem 2.4. (see [3]) *If $c < 0$, then*

$$S_{n-c,c}(x) \geq (1 + 2cx(1 + cx))S_{n,c}(x).$$

If $c > 0$, the inequality is reversed.

Theorem 2.5. (see [3]) *The information potential satisfies the following inequality for all $c \in \mathbb{R}$:*

$$S_{m+n,c}(x) \geq S_{m,c}(x)S_{n,c}(x).$$

Corollary 2.1. (see [3]) *For the Rényi entropy $R_{n,c}(x) := -log(S_{n,c}(x))$ and the Tsallis entropy $T_{n,c}(x) := 1 - S_{n,c}(x)$, we have*

$$\begin{aligned} &R_{m+n,c}(x) \leq R_{m,c}(x) + R_{n,c}(x), \\ &T_{m+n,c}(x) \leq T_{m,c}(x) + T_{n,c}(x) - T_{m,c}(x)T_{n,c}(x). \end{aligned}$$

Now we consider the Shannon entropy

$$H_{n,c}(x) := -\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \log p_{n,k}^{[c]}(x).$$

Theorem 2.6. (see [34]) *The Shannon entropy $H_{n,c}$ and its derivatives verify the following relations*

(i) $H_{n,c}^{(2k+2)}(x) \leq 0,\ c < 0,\ k \geq 0\ x \in \left(0, -\frac{1}{c}\right)$.

$$H_{n,c}^{(2k+1)}(x) = \begin{cases} \geq 0 & x \in (0, -\frac{1}{2c}], \\ \leq 0 & x \in [-\frac{1}{2c}, -\frac{1}{c}). \end{cases}$$

(ii) $H'_{n,0}$ *is completely monotone, i.e.,*

$$(-1)^k H_{n,0}^{(k+1)}(x) \geq 0, \quad k \geq 0, \quad x > 0.$$

(iii) *For $c > 0$, $H'_{n,c}$ is completely monotone.*

(iv) $\log \frac{x}{cx+1} \leq \sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x) \log \frac{k+1}{ck+n} \leq \log \frac{nx+1}{ncx+n},\ x > 0,\ c \geq 0.$

Other properties of $H_{n,0}$ and $R_{n,0}$ can be found in [10]. For other entropies see [38]. Concerning practical applications of the information potential see [5], [39], [11] and the references therein.

3 The Bernstein Basis

If $c = -1$ we are dealing with the functions $p_{n,k}^{[-1]}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, 1 \dots n$, $x \in [0,1]$, i.e., the fundamental Bernstein polynomials. In this case we will use the notation

$$F_n(x) := S_{n,-1}(x) = \sum_{k=0}^{n} \left(\binom{n}{k} x^k (1-x)^{n-k} \right)^2, \quad x \in [0,1].$$

The problem of proving that F_n is a convex function, raised in [17], was solved by Thorsten Neuschel [28] and Geno Nikolov [27], using the following equation

$$F_n(x) = \left(t - \sqrt{t^2 - 1} \right)^n P_n(t),$$

where P_n are the Legendre polynomials, $x \in \left[0, \frac{1}{2}\right)$, $t = \frac{2x^2 - 2x + 1}{1 - 2x} \in [1, +\infty)$. Another solution was given by I. Gavrea and M. Ivan [13]. They found the representation

$$F_n(x) = 4^{-n} \binom{2n}{n} \sum_{k=0}^{n} 4^k \binom{n}{k}^2 \binom{2n}{2k}^{-1} \left(x - \frac{1}{2} \right)^{2k},$$

deduced using Parseval's formula.

The convexity of the function $S_{n,c}$, when $c < 0$ was proved in [32] using (2.1).

Theorem 3.1. (see [32])

(a) *F_n satisfies the following relations:*

$$F_n(x) = \frac{1}{\pi} \int_0^1 \left(t + (1-t)(1-2x)^2 \right)^n \frac{dt}{\sqrt{t(1-t)}},$$

$$2(n+1)F_{n+1}(x) - (2n+1)(1 + (1-2x)^2)F_n(x) + 2n(1-2x)^2 F_{n-1}(x) = 0,$$

$$\begin{aligned}(1-2x)\left(F_{n+1}'(x) - (1 - 2x(1-x))F_n'(x)\right) &= \\ 2(n + 2x(1-x))F_n(x) - 2(n+1)F_{n+1}(x),&\end{aligned}$$

$$F_{n+1}'(x) - (1-2x)^2 F_{n-1}'(x) = 2(1-2x)\left((2n-1)F_{n-1}(x) - (2n+1)F_n(x)\right).$$

(b) *F_n is a solution to the differential equation*

$$x(1-x)(1-2x)y''(x) + (1 + 4(n-1)x(1-x))y'(x) + 2n(1-2x)y(x) = 0.$$

Equivalently, F_n is a polynomial solution to the Heun equation

$$y''(x) + \left(\frac{1}{x} + \frac{1}{x-1} + \frac{-2n}{x - \frac{1}{2}} \right) y'(x) + \frac{-2nx - (-n)}{x(x-1)\left(x - \frac{1}{2}\right)} y(x) = 0,$$

i.e.,

$$F_n(x) = Hl\left(\frac{1}{2}, -n; -2n, 1; 1, 1; x \right).$$

(c) *The following inequality holds:*

$$F_n(x) \leq \frac{1}{\sqrt{1+4(n-1)x(1-x)}}, \quad n \geq 1, x \in [0,1].$$

Theorem 3.2. (see [3]) *The function $F_n(x)$ satisfies the inequalities*

$$F_{n+1}(x) \leq \frac{1+(4n-2)x(1-x)}{1+(4n+2)x(1-x)} F_{n-1}(x),$$

$$F_n(x) \leq \frac{1+4nx(1-x)}{1+(4n+2)x(1-x)} F_{n-1}(x),$$

$$F_n^2(x) \leq F_{n-1}(x)F_{n+1}(x); \; 2F_n(x) \leq F_{n-1}(x) + F_{n+1}(x),$$

$$(1-2x(1-x))F_n(x) \leq F_{n+1}(x) \leq \frac{1+(4n+4)x(1-x)}{1+(4n+6)x(1-x)} F_n(x),$$

$$(1-2x(1-x))^n \leq F_n(x) \leq \sqrt{\frac{1+4x(1-x)}{1+(4n+4)x(1-x)}},$$

$$(1-4x(1-x))^{n/2} \leq F_n(x),$$

$$F_{m+n}(x) \geq F_m(x)F_n(x),$$

for all $n \geq 1, \; x \in [0,1]$.

Corollary 3.1. (see [3]) *The Rényi entropy $R_n(x)$ and the Tsallis entropy $T_n(x)$ corresponding to the binomial distribution with parameters n and x satisfy the inequalities:*

$$R_n(x) - R_{n-1}(x) \geq \log \frac{1+(4n+2)x(1-x)}{1+4nx(1-x)} \geq 0,$$

$$T_n(x) - T_{n-1}(x) \geq \frac{2x(1-x)}{1+4nx(1-x)} (1-T_n(x)) \geq 0.$$

For $c=-1$, Conjecture 2.1 was proved in [33]. A second proof was given in [35]. A simpler proof can be found in [7]. Applications and extensions of these results to the case of a simplex can be found in [33]. A conjecture formulated there was proved in [15].

A conjecture concerning the preservation of log-concavity under the Bernstein operators (see [35]) was settled and extended in [23].

Theorem 3.3. (see [23]) *If $f:[0,1] \to (0.\infty)$ is a log-concave function and $n = 1,2,\ldots,$ then $B_n f$ is log-concave.*

In the context of Approximation Theory I. Gavrea and M. Ivan [14] introduced the operators $L_n : C[0,1] \to C[0,1]$,

$$L_n f = \sum_{k=0}^{n} b_{n,k} f\left(\frac{k}{n}\right), \; n = 1,2,\ldots,$$

where

$$b_{n,k} = \frac{\left(p_{n,k}^{[-1]}\right)^2}{\sum_{i=0}^{n} \left(p_{n,k}^{[-1]}\right)^2}, \; k = 0,1,\ldots,n.$$

Theorem 3.4. (see [14]) *For any $f \in C[0,1]$, the sequence of operators $(L_n f)$ converges uniformly to f on $[0,1]$.*

The Voronovskaja formula for the above sequence of operators (L_n) was established in [19].

Theorem 3.5. (see [19]) *Let $f \in C[0,1]$ be such that the second derivative of f exists and is continuous in a neighborhood of $x \in (0,1)$. Then*

$$\lim_{n\to\infty} [L_n(f,x) - f(x)] = \frac{2x-1}{4} f'(x) + \frac{x(1-x)}{4} f''(x).$$

Similar results involving the powered Szász-Mirakjan and Baskakov bases were obtained in [20] and [21].

4 The Baskakov Basis

In this section, we are concerned with the Baskakov basis

$$\binom{n+k-1}{k} x^k (1+x)^{-n-k}, \quad x \in [0,+\infty), k = 0,1,\ldots,$$

The corresponding information potential (see [32]) is

$$G_n(x) := S_{n,1}(x) = \sum_{k=0}^{\infty} \left(\binom{n+k-1}{k} x^k (1+x)^{-n-k} \right)^2.$$

From the general properties of $S_{n,c}$ it follows that

$$G_n(x) = 4^{1-n} \sum_{k=0}^{n-1} \frac{(2k)!(2n-2k-2)!}{(k!)^2 (n-k-1)!^2} \left(\frac{1}{2x+1} \right)^{2k+1}.$$

Theorem 4.1. (see [32])

(a) *G_n satisfies the following relations:*

$$x(1+x)(1+2x)G_n''(x) + (4(n+1)x(1+x)+1)G_n'(x) + 2n(1+2x)G_n(x) = 0,$$
$$G_n(x) = \frac{1}{\pi} \int_0^1 \left(t + (1-t)(1+2x)^2\right)^{-n} \frac{dt}{\sqrt{t(1-t)}},$$
$$G_n(x) \le (4(n+1)x(1+x)+1)^{-\frac{n}{2(n+1)}},$$
$$G_n(-x) = (1-2x)^{-2n+1} F_{n-1}(x),$$
$$G_n(x) \le \binom{2n-2}{n-1} \frac{(1+x)^{n-1}}{(1+2x)^n}, \quad x \ge 0.$$

(b) *$G_n(-x)$ is a rational solution to the Heun equation*

$$y''(x) + \left(\frac{1}{x} + \frac{1}{x-1} + \frac{2n}{x-\frac{1}{2}} \right) y'(x) + \frac{2nx-n}{x(x-1)\left(x-\frac{1}{2}\right)} y(x) = 0, x < 0,$$

i.e.,

$$G_n(-x) = Hl\left(\frac{1}{2}, n; 2n, 1; 1, 1; x\right).$$

(c) *G_n is a convex function.*

The following relations can be found in [3].

$$n(1+2x)^2G_{n+1}(x) = (2n-1)(1+2x+2x^2)G_n(x) - (n-1)G_{n-1}(x),$$

$$G_n^2 \le G_{n-1}G_{n+1},$$

$$2G_n \le G_{n-1} + G_{n+1},$$

$$G_{n+1}(x) \le \frac{1+(4n-2)x(1+x)}{1+(4n+2)x(1+x)}G_{n-1}(x),$$

$$G_n(x) \le \frac{1+4nx(x+1)}{1+(4n+2)x(x+1)}G_{n-1}(x),$$

$$\frac{1}{1+2x(1+x)}G_n(x) \le G_{n+1}(x) \le \frac{1+(4n+4)x(1+x)}{1+(4n+6)x(1+x)}G_n(x),$$

$$\left(\frac{1}{1+2x(1+x)}\right)^{n-1} \le (1+2x)G_n(x) \le \sqrt{\frac{1+8x(1+x)}{1+(4n+4)x(1+x)}},\ n \ge 1, x \ge 0.$$

5 The Bleimann-Butzer-Hahn Basis

The Bleimann-Butzer-Hahn basis is formed by the functions

$$\binom{n}{k}x^k(1+x)^{-n}, \quad x \in [0,\infty), \quad k = 0,1,\dots,n.$$

The corresponding information potential is given by

$$U_n(x) = \sum_{k=0}^{n}\left(\binom{n}{k}x^k(1+x)^{-n}\right)^2, \quad n \ge 1.$$

It is easy to verify that

$$U_n(x) = F_n\left(\frac{x}{1+x}\right), \quad x \in [0,\infty). \tag{5.1}$$

By virtue of (5.1) the properties of the functions U_n can by deduced from those of F_n. In particular, one has

Theorem 5.1. (see [32])

(a) *U_n is a solution to the differential equation*

$$x(1-x)(1+x)^2y''(x) + (1+x)(1+4nx-x^2)y'(x) + 2n(1-x)y(x) = 0.$$

(b) *The following relations hold true:*

$$U_n(x) = \frac{1}{\pi}\int_0^1\left(t+(1-t)\left(\frac{1-x}{1+x}\right)^2\right)^n\frac{dt}{\sqrt{t(1-t)}},$$

$$U_n(x) = 4^{-n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2n}{2k}^{-1}\left(\frac{x-1}{x+1}\right)^{2k},$$

$$\frac{1}{4^n}\binom{2n}{n} \le U_n(x) \le \frac{x+1}{\sqrt{x^2+(4n-2)x+1}}, \quad x \ge 0, n \ge 1.$$

Theorem 5.2. (see [3])

(i) *The functions $U_n(x)$ satisfy the following relations:*

$$
\begin{aligned}
(n+1)(1+t)^2U_{n+1}(t) &= (2n+1)(t^2+1)U_n(t) - n(1-t)^2U_{n-1}(t),\\
U_n(x) &= 4^{-n}\sum_{k=0}^{n}\binom{2(n-k)}{n-k}\binom{2k}{k}\left(\frac{1-x}{1+x}\right)^{2k},\\
U_n^2 &\le U_{n-1}U_{n+1},\ \ 2U_n \le U_{n-1}+U_{n+1},\\
U_{n+1}(x) &\le \frac{1+4nx+x^2}{1+(4n+4)x+x^2}U_{n-1}(x),\\
U_n(x) &\le \frac{1+(4n+2)x+x^2}{1+(4n+4)x+x^2}U_{n-1}(x),\\
\frac{1+x^2}{(1+x)^2}U_n(x) \le U_{n+1}(x) &\le \frac{1+(4n+6)x+x^2}{1+(4n+8)x+x^2}U_n(x),\\
\left(\frac{1+x^2}{(1+x)^2}\right)^n \le U_n(x) &\le \sqrt{\frac{1+6x+x^2}{1+(4n+6)x+x^2}},\\
U_{m+n}(x) &\ge U_m(x)U_n(x).
\end{aligned}
$$

(ii) *U_n is decreasing on $[0,1]$ and increasing on $[1,\infty)$.*

(iii) *U_n is logarithmically convex on $[0,1]$.*

6 The Meyer-König and Zeller Basis

The Meyer-König and Zeller basis is formed by the functions

$$\binom{n+k}{k}x^k(1-x)^{n+1}, \quad x \in [0,1), k = 0,1,\dots .$$

The corresponding information potential is given by

$$J_n(x) = \sum_{k=0}^{\infty}\left(\binom{n+k}{k}x^k(1-x)^{n+1}\right)^2.$$

It is easy to verify that

$$J_n(x) = G_{n+1}\left(\frac{x}{1-x}\right), \quad x \in [0,1),$$

$$G_n(x) = J_{n-1}\left(\frac{x}{1+x}\right), \quad x \in [0,+\infty).$$

It was proved in [13] that

$$J_n(x) = 4^{-n}\sum_{k=0}^{n}\frac{(2k)!(2n-2k)!}{k!^2(n-k)!^2}\left(\frac{1-x}{1+x}\right)^{2k+1}.$$

Theorem 6.1. (see [32])

(a) *The function J_n satisfies the following relations:*

$$x(1+x)(1-x)^2 J_n''(x) - (1-x)\left(x^2 - 4(n+1)x - 1\right) J_n'(x) + 2(n+1)(1+x)J_n(x) = 0,$$

$$J_n(x) = \frac{1}{\pi}\int_0^1 \left(t + (1-t)\left(\frac{1+x}{1-x}\right)^2\right)^{-n-1} \frac{dt}{\sqrt{t(1-t)}},$$

$$J_n(x) \le \left(\frac{(1-x)^2}{x^2+(4n+6)x+1}\right)^{\frac{n+1}{2(n+2)}}.$$

$$J_n(x) \le \binom{2n}{n}\frac{1-x}{(1+x)^{n+1}}, \quad x \in [0,1).$$

(b) *J_n is a convex function.*

Theorem 6.2. (see [3]) *The functions $J_n(x)$ satisfy the following relations:*

$$(n+1)(1+t)^2 J_{n+1}(t) = (2n+1)(t^2+1)J_n(t) - n(1-t)^2 J_{n-1}(t).$$

$$J_n(x) = 4^{-n}\sum_{k=0}^{n}\binom{2(n-k)}{n-k}\binom{2k}{k}\left(\frac{1-x}{1+x}\right)^{2k+1},$$

$$J_n(x) = \frac{1-x}{1+x}U_n(x).$$

$$J_n^2 \le J_{n-1}J_{n+1},\ 2J_n \le J_{n-1} + J_{n+1},$$

$$J_{n+1}(x) \le \frac{1+4nx+x^2}{1+(4n+4)x+x^2}J_{n-1}(x),$$

$$J_n(x) \le \frac{1+(4n+2)x+x^2}{1+(4n+4)x+x^2}J_{n-1}(x),$$

$$\frac{1+x^2}{(1+x)^2}J_n(x) \le J_{n+1}(x) \le \frac{1+(4n+6)x+x^2}{1+(4n+8)x+x^2}J_n(x),$$

$$\frac{1-x}{1+x}\left(\frac{1+x^2}{(1+x)^2}\right)^n \le J_n(x) \le \frac{1-x}{1+x}\sqrt{\frac{1+6x+x^2}{1+(4n+6)x+x^2}},$$

$$J_{m+n}(x) \ge \frac{1-x}{1+x}J_{m+n}(x) \ge J_m(x)J_n(x).$$

7 The Szász-Mirakjan Basis

The fundamental functions are

$$e^{-nx}\frac{(nx)^k}{k!}, \quad x \in [0,+\infty), k = 0,1,\ldots.$$

Consequently, we will consider the information potential

$$K_n(x) := \sum_{k=0}^{\infty}\left(e^{-nx}\frac{(nx)^k}{k!}\right)^2.$$

Theorem 7.1. (see [32]) *The function K_n satisfies:*

$$xK_n''(x) + (4nx+1)K_n'(x) + 2nK_n(x) = 0,$$

$$K_n(x) = \frac{1}{\pi}\int_{-1}^{1} e^{-2nx(1+t)} \frac{dt}{\sqrt{1-t^2}},$$

$$K_n(x) \le \frac{1}{\sqrt{4nx+1}}, \quad x \ge 0.$$

8 Conclusions and Further Work

This paper has focused on entropies and classical discrete positive linear operators, highlighting their connections with approximation processes and probability distributions. Similar approaches can be applied to newly introduced discrete operators and to continuous distributions associated with integral operators (see [5], [6], [29]). Potential practical applications include information theory, where entropy measures can improve data compression and signal processing; numerical analysis, where operator-related bounds may enhance approximation methods; and statistical physics, where entropies help model uncertainty and complex systems. Open challenges remain in establishing sharper entropy bounds for new operators, extending results to multivariate cases, and exploring connections with special functions such as Heun functions. Overall, these perspectives suggest that the study of entropy in connection with positive linear operators can offer valuable tools for both theoretical analysis and practical applications.

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Conflict of Interest/Competing Interests

The authors declare that they have no conflict of interest and competing interest.

References

[1] U. Abel, W. Gawronski and T. Neuschel, Complete monotonicity and zeros of sums of squared Baskakov functions. Appl. Math. Comput. 258 (2015), 130–137. DOI: 10.1016/j.amc.2015.01.062

[2] A. M. Acu, H. Gonska and I. Raşa, Grüss-type and Ostrowski-type inequalities in approximation theory. Ukrainian Math. J. 63 (2011), no. 6, 843–864. DOI: 10.1007/s11253-011-0548-2

[3] A. M. Acu, A. Măduţa, D. Otrocol and I. Raşa, Inequalities for information potentials and entropies. Mathematics 8 (2020), no. 11, Paper No. 2056. DOI: 10.3390/math8112056

[4] A. M. Acu and M.-D. Rusu, New results concerning Chebyshev-Grüss-type inequalities via discrete oscillations. Appl. Math. Comput. 243 (2014), 585–593. DOI: 10.1016/j.amc.2014.06.008

[5] A. M. Acu, G. Bascanbaz-Tunca and I. Raşa, Information potential for some probability density functions. Appl. Math. Comput. 38 (2021), Paper No. 125578. DOI: 10.1016/j.amc.2020.125578

[6] A. M. Acu, G. Bascanbaz-Tunca and I. Raşa, Bounds for indices of coincidence and entropies. Math. Inequal. Appl. 24 (2021), no. 2, 307–321. DOI: 10.7153/mia-2021-24-22

[7] H. Alzer, Remarks on a convexity theorem of Raşa. Results Math. 75 (2020), no. 1, Paper No. 29. DOI: 10.1007/s00025-020-1156-y

[8] V. Baskakov, An example of a sequence of linear positive operators in the space of continuous functions. Dokl. Akad. Nauk SSSR 113 (1957), 249–251.

[9] E. Berdysheva, Studying Baskakov-Durrmeyer operators and quasi-interpolants via special functions. J. Approx. Theory 149 (2007), no. 2, 131–150. DOI: 10.1016/j.jat.2007.04.009

[10] V. Braiman, A. Malyarenko, Y. Mishura and Y. A. Rudyk, Properties of Shannon and Rényi entropies of the Poisson distribution as the functions of intensity parameter. Nonlinear Anal. Model. Control 29 (2024), no. 4, 802–815. DOI: 10.15388/namc.2024.29.35560

[11] M. Broniatowski and W. Stummer, A Precise Bare simulation approach to the minimization of some distances. I. Foundations. IEEE Trans. Inform. Theory 69 (2023), no. 5, 3062–3120. DOI: 10.1109/TIT.2022.3215496

[12] Digital Library of Mathematical Functions. http://dlmf.nist.gov

[13] I. Gavrea and M. Ivan, On a conjecture concerning the sum of the squared Bernstein polynomials. Appl. Math. Comput. 241 (2014), 70–74. DOI: 10.1016/j.amc.2014.05.003

[14] I. Gavrea and M. Ivan, On a new sequence of positive linear operators related to squared Bernstein polynomials. Positivity 21 (2017), no. 3, 911–917. DOI: 10.1007/s11117-016-0442-0

[15] I. Gavrea and M. Ivan, An answer to an open problem on the multivariate Bernstein polynomials on a simplex. Results Math. 74 (2019), no. 1, Paper No. 13. DOI: 10.1007/s00025-018-0938-y

[16] I. Gavrea and G. Tachev, On the multiplicativity of linear operators. J. Math. Anal. Appl. 408 (2013), no. 1, 203–208. DOI: 10.1016/j.jmaa.2013.05.075

[17] H. Gonska, I. Raşa and M.-D. Rusu, Chebyshev-Grüss-type inequalities via discrete oscillations. Bul. Acad. Stiinte Repub. Mold. Mat. 1 (2014), no. 74, 63–89.

[18] M. Heilmann, Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren. Habilitationsschrift, Universität Dortmund, 1992.

[19] A. Holhoş, Voronovskaya theorem for a sequence of positive linear operators related to squared Bernstein polynomials. Positivity 23 (2019), no. 3, 571–580. DOI: 10.1007/s11117-018-0625-y

[20] A. Holhoş, King-type operators related to squared Szász-Mirakyan basis. Stud. Univ. Babes-Bolyai Math. 65 (2020), no. 2, 279–290. DOI: 10.24193/subbmath.2020.2.09

[21] A. Holhoş, A sequence of positive linear operators related to powered Baskakov basis. Carpathian J. Math. 35 (2019), no. 1, 51–58. DOI: 10.37193/CJM.2019.01.06

[22] M. Hortaşcu, Heun functions and their uses in physics. In: Mathematical Physics, 23–39, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013, pp. 23–39. DOI: 10.1142/9789814417532_0002

[23] A. Komisarski, Log-concavity preserving property of Bernstein operators and Bernstein semigroup. J. Math. Anal. Appl. 489 (2020), no. 1, Paper No. 124107. DOI: 10.1016/j.jmaa.2020.124107

[24] G. Kristensson, Second Order Differential Equations. Special Functions and Their Classification, Springer, 2010. DOI: 10.1007/978-1-4419-7020-6

[25] R. S. Maier, On reducing the Heun equation to the hypergeometric equation. J. Differential Equations 213 (2005), no. 1, 171–203. DOI: 10.1016/j.jde.2004.07.020

[26] R. S. Maier, The 192 solutions of the Heun equation. Math. Comp. 76 (2007), no. 258, 811–843. DOI: 10.1090/S0025-5718-06-01939-9

[27] G. Nikolov, Inequalities for ultraspherical polynomials. Proof of a conjecture of I. Raşa. J. Math. Anal. Appl. 418 (2014), no. 2, 852-860. DOI: 10.1016/j.jmaa.2014.04.022

[28] T. Neuschel, Unpublished manuscript. University of Trier, 2012.

[29] J. C. Principe, Information Theoretic Learning: Rényi's Entropy and Kernel Perspectives. Springer, New York, USA, 2010. DOI: 10.1007/978-1-4419-1570-2

[30] I. Raşa, Special functions associated with positive linear operators. arXiv:1409.1015v2 [math.CA].

[31] I. Raşa and Rényi entropy and Tsallis entropy associated with positive linear operators. arXiv:1412.4971v1 [math.CA].

[32] I. Raşa, Entropies and Heun functions associated with positive linear operators. Appl. Math. Comput. 268 (2015), 422–431. DOI: 10.1016/j.amc.2015.06.085

[33] I. Raşa, Convexity properties of some entropies. Results Math. 73 (2018), no 3, Paper No. 105. DOI: 10.1007/s00025-018-0868-8

[34] I. Raşa, Complete monotonicity of some entropies. Period. Math. Hungar. 75 (2017), no. 2, 159–166. DOI: 10.1007/s10998-016-0177-5

[35] I. Raşa, Convexity properties of some entropies (II), Results Math. 74 (2019), no. 4, Paper No. 154. DOI: 10.1007/s00025-019-1081-0

[36] A. Rényi, On measures of entropy and information. University of California Press, Berkeley-Los Angeles, Calif., 1960, pp. 547–561.

[37] A. Ronveaux, Heun's Differential Equations, Oxford Univ. Press, 1995.

[38] R.-C. Sfetcu, E.-G. Robe-Voinea and F. Serban, An alternate measure of the cumulative residual Sharma-Taneja-Mittal entropy. An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 33 (2025), no. 2, 125–142.

[39] Y. Sun and Z. Sun, Generating probability distributions on intervals and spheres: convex decomposition. Comput. Math. Appl. 154, 2024, 12–23. DOI: 10.1016/j.camwa.2023.11.006

[40] G. Tachev, On multiplicativity of the Bernstein operator. Comput. Math. Appl. 62 (2011), no. 8, 3236–3240. DOI: 10.1016/j.camwa.2011.08.038

[41] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics. J. Stat. Phys. 52 (1988), no. 1-2, 479–487. DOI: 10.1007/BF01016429

[42] C. Tsallis, The nonadditive entropy S_q and its applications in physics and elsewhere: some remarks, Entropy 13 (2011), no. 10, 1765–1804. DOI: 10.3390/e13101765

[43] M. Wagner, Quasi-Interpolaten zu genuinen Baskakov-Durrmeyer-Typ Operatoren, Shaker Verlag, Aachen, 2013.

[44] D. Xu and D. Erdogmuns, Rényi entropy, divergence and their nonparametric estimators. In: Information Theoretic Learning: Rényi Entropy and Kernel Perspectives, Ed. J.C. Principe, Springer Science + Business Media, LLC 2010, pp. 47–102.