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Research Article

Bilateral Generating Relations Involving Generalized k-Cesàro Polynomials and k-Lauricella Functions

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Abstract: In this paper, we introduce a new generalizad class of Lauricella functions by means of the k-special function theory and investigate some of their fundamental properties. We derive several families of bilateral generating relations for these functions. The obtained results establish new links between these families and generalize a variety of known generating function identities. Several special cases are also discussed. In particular, it is shown that the classical generating relations associated with generalized Cesàro polynomials and Lauricella functions are recovered when $k = 1$.

Keywords: Generalized k-Cesàro polynomials, generating functions, k-Lauricella functions.

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1 Introduction

Special functions occupy a central position in both pure and applied mathematics due to their numerous applications in fields such as physics, chemistry, engineering, and biological sciences. Their rich analytical structures and wide applicability have made them indispensable tools in the investigation of a variety of mathematical models and scientific phenomena. As a result, the search for new generalizations and extensions of classical special functions remains an active area of research.

Many classical special functions are defined through infinite series, integral representations, or product expansions. In recent years, significant efforts have been devoted to extending these functions in order to obtain more general and flexible mathematical frameworks. In particular, the theory of k-special functions has emerged as an important branch of research, leading to k-analogues of several well-known functions, including the Gamma, Beta, and hypergeometric functions. Motivated by these developments and by the works reported in [1, 2], we undertake the present investigation.

The polynomials $g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k}, x\right)$, defined by the generating function

$$\sum_{n=0}^{\infty} g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k}, x\right) t^n = (1-t)^{-\frac{s}{k}-1}(1-kxt)^{-\frac{\lambda}{k}}, \tag{1.1}$$

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constitute a natural k-extension of the classical generalized Cesàro polynomials and appear in several problems arising in mathematical statistics (cf. Çin et al. [3]).

Clearly, the generating function (1.1) yields the following explicit representation ([3], p. 69, Eq. (15)):

$$g_{n,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},x\right)=\sum_{h=0}^{n}\left(\frac{s}{k}+1\right)_{n-h}\frac{(\lambda)_{h,k}\ x^{h}}{(n-h)!h!}. \tag{1.2}$$

On the other hand, let $\mathfrak{x}\in\mathbb{C}$, $k\in\mathbb{R}^{+}$ and $\mathfrak{n}\in\mathbb{N}$. Then, the Pochhammer k-symbol is defined in [4] by

$$(\mathfrak{x})_{\mathfrak{n},k}=\mathfrak{x}\left(\mathfrak{x}+k\right)\left(\mathfrak{x}+2k\right)...\left(\mathfrak{x}+\left(\mathfrak{n}-1\right)k\right).$$

In particular, we denote $(\mathfrak{x})_{0,k}:=1$.

Furthermore, these polynomials satisfy the following generating relation [3]:

$$\sum_{h=0}^{\infty}\left(\begin{array}{c}h+p\\h\end{array}\right)g_{h+p,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},x\right)t^{h}=(1-t)^{-\frac{s}{k}-p-1}(1-kxt)^{-\frac{\lambda}{k}}g_{p,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},\frac{x(1-t)}{1-kxt}\right), \tag{1.3}$$

where $p=0,1,2,....$

Definition 1.1. [4] *For $\mathfrak{x}\in\mathbb{C}$ and $k\in\mathbb{R}^{+}$, the integral representation of k-gamma function Γ_k is defined by*

$$\Gamma_{k}(\mathfrak{x})=\int_{0}^{\infty}t^{\mathfrak{x}-1}e^{-\frac{t^{k}}{k}}dt,$$

where $Re(\mathfrak{x})>0$.

Lemma 1.1. [5] *If $\varpi\in\mathbb{C}$ and $m,p\in\mathbb{N}$ then for $k\in\mathbb{R}^{+}$, we have*

$$\begin{aligned}(\varpi)_{p,k}&=\frac{\Gamma_{k}(\varpi+pk)}{\Gamma_{k}(\varpi)},\\(\varpi)_{p,k}&=k^{p}\left(\frac{\varpi}{k}\right)_{p}, &(1.4)\\(\varpi)_{m+p,k}&=(\varpi)_{m,k}(\varpi+mk)_{p,k}, &(1.5)\end{aligned}$$

where $(\varpi)_p$ and $(\varpi)_{p,k}$ denote the Pochhammer symbol and Pochhammer k-symbol, respectively.

Motivated by recent advances in k-special function theory, this paper explores bilateral generating relations involving generalized k-Cesàro polynomials and k-Srivastava–Daoust functions. Several auxiliary definitions are introduced, and a collection of new generating identities is derived, leading to further connections between these families of functions.

2 Fundamental Definitions and Properties

This section provides the preliminary framework for our investigation. We introduce the notation and auxiliary results that will be used throughout the paper and review the corresponding definition together with several properties that are essential for the derivation and proof of the subsequent results.

Definition 2.1. *For $n \in \mathbb{N}$ and $k \in \mathbb{R}^{+}$, the generalized k-Lauricella function is introduced through the following series representation:*

$$F_{(C,k):\left(D^{(1)},k\right);\ldots;\left(D^{(n)},k\right)}^{(A,k):\left(B^{(1)},k\right);\ldots;\left(B^{(n)},k\right)}\left(\begin{array}{l}\left[(a):\varphi^{(1)},\ldots,\varphi^{(n)}\right]: \\ \\ \left[(c):\chi^{(1)},\ldots,\chi^{(n)}\right]: \\ \end{array}\right. \tag{2.1}$$

$$\left.\begin{array}{r}\left[\left(b^{(1)}\right):\Psi^{(1)}\right];\ldots;\left[\left(b^{(n)}\right):\Psi^{(n)}\right]; \\ \\ \left[\left(d^{(1)}\right):\zeta^{(1)}\right];\ldots;\left[\left(d^{(n)}\right):\zeta^{(n)}\right]; \\ \end{array}\right. z_1,\ldots,z_n\Bigg)$$

$$=\sum_{s_1,\ldots,s_n=0}^{\infty}\Omega_k\left(s_1,\ldots,s_n\right)\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!},$$

where, for convenience,

$$\Omega_k\left(s_1,\ldots,s_n\right):=\frac{\prod_{j=1}^{A}\left(a_j\right)_{s_1\varphi_j^{(1)}+\ldots+s_n\varphi_j^{(n)},k}}{\prod_{j=1}^{C}\left(c_j\right)_{s_1\chi_j^{(1)}+\ldots+s_n\chi_j^{(n)},k}}\frac{\prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{s_1\Psi_j^{(1)},k}}{\prod_{j=1}^{D^{(1)}}\left(d_j^{(1)}\right)_{s_1\zeta_j^{(1)},k}}\cdots\frac{\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{s_n\Psi_j^{(n)},k}}{\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{s_n\zeta_j^{(n)},k}};$$

the coefficients

$$\varphi_j^{(s)}\quad\left(j=1,\ldots,A;\ s=1,\ldots,n\right)\ \text{and}\ \ \Psi_j^{(s)}\quad\left(j=1,\ldots,B^{(s)};\ s=1,\ldots,n\right),$$
$$\chi_j^{(s)}\quad\left(j=1,\ldots,C;\ s=1,\ldots,n\right)\ \text{and}\ \ \zeta_j^{(s)}\quad\left(j=1,\ldots,D^{(s)};\ s=1,\ldots,n\right)$$

are real constants and $\left(b_{B^{(s)}}^{(s)}\right)$ abbreviates the array of $B^{(s)}$ parameters $b_j^{(s)}$ $\left(j=1,\ldots,B^{(s)};\ s=1,\ldots,n\right)$ for all $s\in\{1,\ldots,n\}$.

Remark 2.1. *The following k-generalization of the classical Lauricella functions is obtained by appropriately choosing the parameters in equation (2.1).*

1. Specifically, if we set

$$A=B^{(1)}=\ldots=B^{(n)}=D^{(1)}=\ldots=D^{(n)}=1,\ C=0$$

and

$$\varphi^{(1)}=\ldots=\varphi^{(n)}=\Psi^{(1)}=\ldots=\Psi^{(n)}=\zeta^{(1)}=\ldots=\zeta^{(n)}=1$$

in expression (2.1), we obtain

$$F_{A,k}^{(n)}\left[a,b_1,\ldots,b_n;c_1,\ldots,c_n;z_1,\ldots,z_n\right] \tag{2.2}$$
$$=\sum_{s_1,\ldots,s_n=0}^{\infty}\frac{(a)_{s_1+\ldots+s_n,k}\left(b_1\right)_{s_1,k}\cdots\left(b_n\right)_{s_n,k}}{\left(c_1\right)_{s_1,k}\cdots\left(c_n\right)_{s_n,k}}\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}$$

$$\left(|z_1|+\ldots+|z_n|<\frac{1}{k}\right).$$

2. Specifically, if we set

$$C=1,\ B^{(1)}=\ldots=B^{(n)}=2,\ A=D^{(1)}=\ldots=D^{(n)}=0$$

and

$$\Psi^{(1)}=\ldots=\Psi^{(n)}=\chi^{(1)}=\ldots=\chi^{(n)}=1$$

in expression (2.1), we obtain

$$\begin{aligned}&F_{B,k}^{(n)}\left[a_1,\ldots,a_n,b_1,\ldots,b_n;c;z_1,\ldots,z_n\right] \\ &=\sum_{s_1,\ldots,s_n=0}^{\infty}\frac{(a_1)_{s_1,k}\cdots(a_n)_{s_n,k}\,(b_1)_{s_1,k}\cdots(b_n)_{s_n,k}}{(c)_{s_1+\ldots+s_n,k}}\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}\end{aligned} \tag{2.3}$$

$$\left(\max\left\{|z_1|,\ldots,|z_n|\right\}<\frac{1}{k}\right).$$

3. Specifically, if we set

$$C=B^{(1)}=\ldots=B^{(n)}=0,\ D^{(1)}=\ldots=D^{(n)}=1,\ A=2$$

and

$$\varphi^{(1)}=\ldots=\varphi^{(n)}=\zeta^{(1)}=\ldots=\zeta^{(n)}=1$$

in expression (2.1), we obtain

$$\begin{aligned}&F_{C,k}^{(n)}\left[a,b;c_1,\ldots,c_n;z_1,\ldots,z_n\right] \\ &=\sum_{s_1,\ldots,s_n=0}^{\infty}\frac{(a)_{s_1+\ldots+s_n,k}\,(b)_{s_1+\ldots+s_n,k}}{(c_1)_{s_1,k}\cdots(c_n)_{s_n,k}}\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}\end{aligned} \tag{2.4}$$

$$\left(\sqrt{|z_1|}+\ldots+\sqrt{|z_n|}<\frac{1}{\sqrt{k}}\right).$$

4. Specifically, if we set

$$A=C=1,\ B^{(1)}=\ldots=B^{(n)}=1,\ D^{(1)}=\ldots=D^{(n)}=0$$

and

$$\varphi^{(1)}=\ldots=\varphi^{(n)}=\Psi^{(1)}=\ldots=\Psi^{(n)}=\chi^{(1)}=\ldots=\chi^{(n)}=1$$

in expression (2.1), we obtain

$$\begin{aligned}&F_{D,k}^{(n)}\left[a,b_1,\ldots,b_n;c;z_1,\ldots,z_n\right] \\ &=\sum_{s_1,\ldots,s_n=0}^{\infty}\frac{(a)_{s_1+\ldots+s_n,k}\,(b_1)_{s_1,k}\cdots(b_n)_{s_n,k}}{(c)_{s_1+\ldots+s_n,k}}\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}\end{aligned} \tag{2.5}$$

$$\left(\max\left\{|z_1|,\ldots,|z_n|\right\}<\frac{1}{k}\right).$$

It should be noted here that the special cases given in (2.2)–(2.5) were first defined in the master's thesis [6].

Remark 2.2. *By specializing the parameter n to 2 in (2.2)–(2.5), one obtains the k-Appell functions $F_{1,k}$, $F_{2,k}$, $F_{3,k}$ and $F_{4,k}$ respectively [7, 8].*

Remark 2.3. *It is easily verified that Eqs. (2.2)–(2.5) reduce to the generalized Lauricella functions when $k=1$ [9].*

3 The Main Generating Functions

Motivated by the construction of generalized generating relations, we introduce the family of functions $\phi_{n,k}(u_1;u_2,\ldots,u_r)$ associated with a suitably restricted coefficient sequence

$$\{\Omega_k(f(m_1,\ldots,m_r);m_2,\ldots,m_r)\}_{m_1,m_2,\ldots,m_r\in\mathbb{N}_0}.$$

These functions are defined by the multiple series

$$\begin{aligned}\phi_{n,k}(u_1;u_2,\ldots,u_r) &:= \sum_{m_1=0}^{n}\sum_{m_2,\ldots,m_r=0}^{\infty}\frac{(-nk)_{m_1,k}\,((b))_{m_1\Psi,k}}{((d))_{m_1\zeta,k}} \\ &\quad\times\Omega_k(f(m_1,\ldots,m_r);m_2,\ldots,m_r)\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_r^{m_r}}{m_r!},\end{aligned} \tag{3.1}$$

where for convenience

$$((b))_{m_1\Psi,k}=\prod_{j=1}^{B}(b_j)_{m_1\Psi_j,k}\quad\text{and}\quad((d))_{m_1\zeta,k}=\prod_{j=1}^{D}(d_j)_{m_1\zeta_j,k}.$$

The following theorem establishes several bilateral generating relations for our generalized functions.

Theorem 3.1. *The following bilateral generating function identities hold:*

$$\begin{aligned}&\sum_{n=0}^{\infty}g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\phi_{n,k}(u_1;u_2,\ldots,u_r)\,t^n \\ &=(1-t)^{-\frac{s}{k}-1}(1-kxt)^{-\frac{\lambda}{k}}\sum_{m_1,p,m_2,\ldots,m_r=0}^{\infty}\frac{((b))_{(m_1+p)\Psi,k}\,(s+k)_{m_1,k}\,(\lambda)_{p,k}}{((d))_{(m_1+p)\zeta,k}} \\ &\quad\times\Omega_k(f((m_1+p),m_2,\ldots,m_r);m_2,\ldots,m_r)\frac{\left(\frac{u_1t}{t-1}\right)^{m_1}}{m_1!}\frac{\left(\frac{ku_1xt}{kxt-1}\right)^{p}}{p!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_r^{m_r}}{m_r!},\end{aligned} \tag{3.2}$$

where $\phi_{n,k}(u_1;u_2,\ldots,u_r)$ *is given by* (3.1).

Proof. We first recall the following double series equality [10, pp. 56-57, Section 37]

$$\sum_{h=0}^{\infty}\sum_{p=0}^{h}V(p,h)=\sum_{h=0}^{\infty}\sum_{p=0}^{\infty}V(p,h+p). \tag{3.3}$$

If Substituting (3.1) into the left-hand side of (3.2), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\phi_{n,k}(u_1;u_2,\ldots,u_r)\,t^n \\ &=\sum_{n=0}^{\infty}g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\sum_{m_1=0}^{n}\sum_{m_2,\ldots,m_r=0}^{\infty}\frac{n!\;((b))_{m_1\Psi,k}}{(n-m_1)!m_1!\,((d))_{m_1\zeta,k}} \\ &\quad\times\Omega_k(f(m_1,\ldots,m_r);m_2,\ldots,m_r)(-ku_1)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_r^{m_r}}{m_r!}t^n.\end{aligned} \tag{3.4}$$

If (3.3) is used in (3.4), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_{n,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},x\right)\phi_{n,k}\left(u_1;u_2,\ldots,u_r\right)t^n\\
&=\sum_{n=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2,\ldots,m_r=0}^{\infty} g_{n+m_1,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},x\right)\frac{(n+m_1)!\ \left((b)\right)_{m_1\Psi,k}}{n!m_1!\left((d)\right)_{m_1\zeta,k}}\\
&\quad\times\Omega_k\left(f\left(m_1,\ldots,m_r\right);m_2,\ldots,m_r\right)\left(-ku_1\right)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_r^{m_r}}{m_r!}t^{n+m_1}\\
&=\sum_{m_1,\ldots,m_r=0}^{\infty}\left[\sum_{n=0}^{\infty}\left(\begin{array}{c} n+m_1\\ n\end{array}\right)g_{n+m_1,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},x\right)t^n\right]\frac{\left((b)\right)_{m_1\Psi,k}}{\left((d)\right)_{m_1\zeta,k}} \qquad (3.5)\\
&\times\Omega_k\left(f\left(m_1,\ldots,m_r\right);m_2,\ldots,m_r\right)\left(-ku_1t\right)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_r^{m_r}}{m_r!}.
\end{aligned}$$

If (1.3) is used in (3.5), we attain

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_{n,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},x\right)\phi_{n,k}\left(u_1;u_2,\ldots,u_r\right)t^n\\
&=\left(1-t\right)^{-\frac{s}{k}-1}\left(1-kxt\right)^{-\frac{\lambda}{k}}\sum_{m_1,\ldots,m_r=0}^{\infty} g_{m_1,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},\frac{x\left(1-t\right)}{1-kxt}\right)\frac{\left((b)\right)_{m_1\Psi,k}}{\left((d)\right)_{m_1\zeta,k}} \qquad (3.6)\\
&\quad\times\Omega_k\left(f\left(m_1,\ldots,m_r\right);m_2,\ldots,m_r\right)\left(\frac{ku_1t}{t-1}\right)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_r^{m_r}}{m_r!}.
\end{aligned}$$

If (1.2) is applied to (3.6), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_{n,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},x\right)\phi_{n,k}\left(u_1;u_2,\ldots,u_r\right)t^n\\
&=\left(1-t\right)^{-\frac{s}{k}-1}\left(1-kxt\right)^{-\frac{\lambda}{k}}\sum_{m_1,\ldots,m_r=0}^{\infty}\sum_{p=0}^{m_1}\left(\frac{s}{k}+1\right)_{m_1-p}\frac{(\lambda)_{p,k}\left(\frac{x(1-t)}{1-kxt}\right)^p}{(m_1-p)!p!}\frac{\left((b)\right)_{m_1\Psi,k}}{\left((d)\right)_{m_1\zeta,k}} \qquad (3.7)\\
&\quad\times\Omega_k\left(f\left(m_1,\ldots,m_r\right);m_2,\ldots,m_r\right)\left(\frac{ku_1t}{t-1}\right)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_r^{m_r}}{m_r!}.
\end{aligned}$$

If (3.3) and (1.4) are applied to (3.7) respectively, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_{n,k}^{(\frac{s}{k})}\left(\frac{\lambda}{k},x\right)\phi_{n,k}\left(u_1;u_2,\ldots,u_r\right)t^n\\
&=\left(1-t\right)^{-\frac{s}{k}-1}\left(1-kxt\right)^{-\frac{\lambda}{k}}\sum_{m_1,p,m_2,\ldots,m_r=0}^{\infty}\frac{\left((b)\right)_{(m_1+p)\Psi,k}\left(s+k\right)_{m_1,k}\left(\lambda\right)_{p,k}}{\left((d)\right)_{(m_1+p)\zeta,k}}\\
&\quad\times\Omega_k\left(f\left(m_1+p,m_2,\ldots,m_r\right);m_2,\ldots,m_r\right)\frac{\left(\frac{u_1t}{t-1}\right)^{m_1}}{m_1!}\frac{\left(\frac{ku_1xt}{kxt-1}\right)^p}{p!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_r^{m_r}}{m_r!}.
\end{aligned}$$

Thus the proof of (3.2) is completed. □

A variety of interesting consequences follow from Theorem 3.1 through appropriate selections of the multiple sequence $\Omega_k\left(m_1,m_2,\ldots,m_r\right)$. In particular, several families of bilateral generating relations

associated with the generalized k-Cesàro polynomials and the generalized k-Lauricella functions are obtained.

1) By letting

$$\begin{aligned}&\Omega_k\left(f\left(m_1,m_2,\ldots,m_r\right);m_2,\ldots,m_r\right)\\&=\frac{\prod_{j=1}^{A}\left(a_j\right)_{m_1\varphi_j^{(1)}+\ldots+m_r\varphi_j^{(r)},k}}{\prod_{j=1}^{E}\left(c_j\right)_{m_1\chi_j^{(1)}+\ldots+m_r\chi_j^{(r)},k}}\frac{\prod_{j=1}^{B^{(2)}}\left(b_j^{(2)}\right)_{m_2\Psi_j^{(2)},k}}{\prod_{j=1}^{D^{(2)}}\left(d_j^{(2)}\right)_{m_2\zeta_j^{(2)},k}}\cdots\frac{\prod_{j=1}^{B^{(r)}}\left(b_j^{(r)}\right)_{m_r\Psi_j^{(r)},k}}{\prod_{j=1}^{D^{(r)}}\left(d_j^{(r)}\right)_{m_r\zeta_j^{(r)},k}}\end{aligned}$$

in Theorem 3.1, we get the following results.

Corollary 3.1. *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty}g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\\&\quad\times F_{(E,k):\quad(D,k);\left(D^{(2)},k\right);\ldots;\left(D^{(r)},k\right)}^{(A,k):(B+1,k);\left(B^{(2)},k\right);\ldots;\left(B^{(r)},k\right)}\left(\begin{array}{l}\left[(a):\varphi^{(1)},\ldots,\varphi^{(r)}\right]:\quad[-nk:1],\quad[(b):\Psi];\\\left[(c):\chi^{(1)},\ldots,\chi^{(r)}\right]:\qquad-,\qquad[(d):\zeta];\\\qquad\left[\left(b^{(2)}\right):\Psi^{(2)}\right];\ldots;\left[\left(b^{(r)}\right):\Psi^{(r)}\right];\\\qquad\left[\left(d^{(2)}\right):\zeta^{(2)}\right];\ldots;\left[\left(d^{(r)}\right):\zeta^{(r)}\right];\end{array}u_1,\ldots,u_r\right)t^n\\&=(1-t)^{-\frac{s}{k}-1}(1-kxt)^{-\frac{\lambda}{k}}\\&\quad\times F_{(E+D,k):\quad0\ ;\quad0;\ \left(D^{(2)},k\right);\ldots;\left(D^{(r)},k\right)}^{(A+B,k):(1,k);(1,k);\left(B^{(2)},k\right);\ldots;\left(B^{(r)},k\right)}\left(\begin{array}{l}\left[(e):\xi^{(1)},\ldots,\xi^{(r+1)}\right]:\quad[s+k:1];\quad[\lambda:1];\\\left[(f):\delta^{(1)},\ldots,\delta^{(r+1)}\right]:\qquad-;\qquad-;\\\qquad\left[\left(b^{(2)}\right):\Psi^{(2)}\right];\ldots;\left[\left(b^{(r)}\right):\Psi^{(r)}\right];\\\qquad\left[\left(d^{(2)}\right):\zeta^{(2)}\right];\ldots;\left[\left(d^{(r)}\right):\zeta^{(r)}\right];\end{array}\left(\frac{u_1t}{t-1}\right),\left(\frac{ku_1xt}{kxt-1}\right),u_2,\ldots,u_r\right),\end{aligned}$$

where the parameters e_j, f_j and the coefficients $\xi_j^{(r)}$, $\delta_j^{(r)}$ are given by

$$e_j=\begin{cases}a_j & (1\leq j\leq A)\\ b_{j-A} & (A<j\leq A+B),\end{cases}$$

$$f_j=\begin{cases}c_j & (1\leq j\leq E)\\ d_{j-E} & (E<j\leq E+D),\end{cases}$$

$$\xi_j^{(v)}=\begin{cases}\varphi_j^{(1)} & (1\leq j\leq A;\ 1\leq v\leq 2)\\ \varphi_j^{(v-1)} & (1\leq j\leq A;\ 2<v\leq r+1)\\ \Psi_{j-A} & (A<j\leq A+B;\ 1\leq v\leq 2)\\ 0 & (A<j\leq A+B;\ 2<v\leq r+1)\end{cases}$$

and

$$\delta_j^{(v)} = \begin{cases} \chi_j^{(1)} & (1 \leq j \leq E;\ 1 \leq v \leq 2) \\ \chi_j^{(v-1)} & (1 \leq j \leq E;\ 2 < v \leq r+1) \\ \zeta_{j-E} & (E < j \leq E+D;\ 1 \leq v \leq 2) \\ 0 & (E < j \leq E+D;\ 2 < v \leq r+1) \end{cases}$$

respectively.

2) Upon setting

$$\Omega_k\left(f\left(m_1, m_2, \ldots, m_r\right); m_2, \ldots, m_r\right) = \frac{(a)_{m_1+\ldots+m_r,k}\,(b_2)_{m_2,k} \cdots (b_r)_{m_r,k}}{(c_1)_{m_1,k} \cdots (c_r)_{m_r,k}}$$

and

$$\Psi = \zeta = 0 \quad (\text{that is, } \Psi_1 = \cdots = \Psi_B = \zeta_1 = \cdots = \zeta_D = 0)$$

in Theorem 3.1, we attain the following result.

Corollary 3.2. *The following bilateral generating function holds true:*

$$\begin{aligned} &\sum_{n=0}^{\infty} g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k}, x\right) F_{A,k}^{(r)}\left[a, -nk, b_2, \ldots, b_r; c_1, \ldots, c_r; u_1, \ldots, u_r\right] t^n \\ &= (1-t)^{-\frac{s}{k}-1}\,(1-kxt)^{-\frac{\lambda}{k}} \\ &\quad \times F_{(1,k):\,0\,;\,0\,;\,(1,k);\ldots;(1,k)}^{(1,k):(1,k);(1,k);(1,k);\ldots;(1,k)} \left(\begin{array}{lll} [(a):1,\ldots,1]: & [s+k:1]; & [\lambda:1]; \\ \left[(c_1):\psi^{(1)},\ldots,\psi^{(r+1)}\right]: & -; & -; \\ [b_2:1]; \ldots; [b_r:1]; & \left(\frac{u_1 t}{t-1}\right), \left(\frac{k u_1 x t}{kxt-1}\right), u_2, \ldots, u_r & \\ [c_2:1]; \ldots; [c_r:1]; & & \end{array} \right), \end{aligned}$$

where the coefficients $\psi^{(\tau)}$ *are given by*

$$\psi^{(\tau)} = \begin{cases} 1 & (1 \leq \tau \leq 2) \\ 0 & (2 < \tau \leq r+1)\,. \end{cases}$$

3) If we put

$$\Omega_k\left(f\left(m_1, m_2, \ldots, m_r\right); m_2, \ldots, m_r\right) = \frac{\left(a_1^{(1)}\right)_{m_2,k} \cdots \left(a_1^{(r-1)}\right)_{m_r,k} \left(a_2^{(1)}\right)_{m_2,k} \cdots \left(a_2^{(r-1)}\right)_{m_r,k}}{(c)_{m_1+\ldots+m_r,k}}$$

and

$$B = 1,\ b_1 = b,\ \Psi_1 = 1,\ \text{and } \zeta = 0$$

in Theorem 3.1, we obtain Corollary 3.3 below.

Corollary 3.3. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) F_{B,k}^{(r)}\left[-nk,a_1^{(1)},\ldots,a_1^{(r-1)},b,a_2^{(1)},\ldots,a_2^{(r-1)};c;u_1,\ldots,u_r\right]t^n$$
$$=(1-t)^{-\frac{s}{k}-1}(1-kxt)^{-\frac{\lambda}{k}}$$
$$\times F_{(1,k):\,0\,;\;0;\;\;0;\ldots;0}^{(1,k):(1,k);(1,k);(2,k);\ldots;(2,k)}\left(\begin{array}{lll}\left[(b):\theta^{(1)},\ldots,\theta^{(r+1)}\right]: & [s+k:1]; & [\lambda:1];\\ & & \\ [(c):1,\ldots,1]: & -; & -;\\ \left[a^{(1)}:1\right];\ldots;\left[a^{(r-1)}:1\right]; & & \\ & \left(\frac{u_1t}{t-1}\right),\left(\frac{ku_1xt}{kxt-1}\right),u_2,\ldots,u_r & \\ -\;; & \ldots; & -\;;\end{array}\right),$$

where the coefficients $\theta^{(\tau)}$ are given by

$$\theta^{(\tau)}=\begin{cases}1 & (1\leq\tau\leq 2)\\ 0 & (2<\tau\leq r+1)\,.\end{cases}$$

4) By letting

$$\Omega_k\left(f\left(m_1,m_2,\ldots,m_r\right);m_2,\ldots,m_r\right)=\frac{(a)_{m_1+\ldots+m_r,k}\,(b_2)_{m_2,k}\cdots(b_r)_{m_r,k}}{(c)_{m_1+\ldots+m_r,k}}$$

and

$$\Psi=\zeta=0$$

in Theorem 3.1, we obtain the following result.

Corollary 3.4. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) F_{D,k}^{(r)}\left[a,-nk,b_2,\ldots,b_r;c;u_1,\ldots,u_r\right]t^n$$
$$=(1-t)^{-\frac{s}{k}-1}(1-kxt)^{-\frac{\lambda}{k}}F_{D,k}^{(r+1)}\left[a,s+k,\lambda,b_2,\ldots,b_r;c;\frac{u_1t}{t-1},\frac{ku_1xt}{kxt-1},u_2,\ldots,u_r\right].$$

Theorem 3.2. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}\,(a_2)_{n,k}}{(s)_{n,k}\,(\lambda)_{n,k}}g_{n,k}^{\left(\frac{-s-nk}{k}\right)}\left(\frac{-\lambda-nk+k}{k},x\right) \tag{3.8}$$
$$\times F_{C,k}^{(r)}\left[a_1+nk,a_2+nk;c_1,\ldots,c_r;u_1,\ldots,u_r\right]t^n$$
$$=F_{0\;:(1,k);(1,k);\ldots;(1,k)}^{(2,k):\;\;0\;\;;\;\;0\;\;;\ldots;\;\;0}\left(\begin{array}{lllll}[(a):1,\ldots,1]: & -\;; & -\;; & -\;; & \ldots\;;\\ & & & & \\ [-:-,\ldots,-]: & [\lambda:1]; & [s:1]; & [c_1:1]; & \ldots\;;\\ & & & -\;; & \\ & & & -k^{-1}t,-xt,u_1,\ldots,u_r & \\ & & & [c_r:1]; & \end{array}\right),$$

where

$$((a))_n=\prod_{j=1}^{A}(a_j)_n\,.$$

Proof. If (1.2) and (2.4) are used on the left side of the Theorem, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}\,(a_2)_{n,k}}{(s)_{n,k}\,(\lambda)_{n,k}}g_{n,k}^{(\frac{-s-nk}{k})}\left(\frac{-\lambda-nk+k}{k},x\right)F_{C,k}^{(r)}\left[a_1+nk,a_2+nk;c_1,\ldots,c_r;u_1,\ldots,u_r\right]t^n\\
&=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(a_1)_{n,k}\,(a_2)_{n,k}}{(s)_{n,k}\,(\lambda)_{n,k}}\left(\frac{-s-nk}{k}+1\right)_{n-p}\frac{(-\lambda-nk+k)_{p,k}\,x^p}{(n-p)!p!}\qquad(3.9)\\
&\quad\times\frac{(a_1+nk)_{m_1+\cdots+m_r,k}\,(a_2+nk)_{m_1+\cdots+m_r,k}}{(c_1)_{m_1,k}\cdots(c_r)_{m_r,k}}\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_r^{m_r}}{m_r!}t^n.
\end{aligned}$$

Applying (3.3), (1.4), and (1.5) to (3.9), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(a_1)_{n,k}\,(a_2)_{n,k}}{(s)_{n,k}\,(\lambda)_{n,k}}g_{n,k}^{(\frac{-s-nk}{k})}\left(\frac{-\lambda-nk+k}{k},x\right)F_{C,k}^{(r)}\left[a_1+nk,a_2+nk;c_1,\ldots,c_r;u_1,\ldots,u_r\right]t^n\\
&=\sum_{n,p,m_1,\ldots,m_r=0}^{\infty}\frac{(a_1)_{n+p+m_1+\cdots+m_r,k}\,(a_2)_{n+p+m_1+\cdots+m_r,k}}{(\lambda)_{n,k}\,(s)_{p,k}\,(c_1)_{m_1,k}\cdots(c_r)_{m_r,k}}\frac{\left(-k^{-1}t\right)^n}{n!}\frac{(-xt)^p}{p!}\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_r^{m_r}}{m_r!}\\
&=F\begin{matrix}(2,k):\ 0\ ;\ 0\ ;\ldots;\ 0\\ 0\ :(1,k);(1,k);\ldots;(1,k)\end{matrix}\left(\begin{matrix}[(a):1,\ldots,1]:&-\ ;&-\ ;&-\ ;&\ldots\ ;\\ [-:-,\ldots,-]:&[\lambda:1]\,;&[s:1]\,;&[c_1:1]\,;&\ldots\ ;\\ &&&&-\ ;\\ &&&&[c_r:1]\,;\end{matrix}\right.\\
&\qquad\left.\begin{matrix}\\ \\ -k^{-1}t,-xt,u_1,\ldots,u_r\end{matrix}\right).
\end{aligned}$$

Thus the proof of statement (3.8) is completed. □

4 Conclusion

In this paper, several families of bilateral generating function identities involving the generalized k-Cesàro polynomials and the generalized k-Lauricella functions have been established. The results obtained reveal new connections between these two important classes of k-special functions and further enrich the theory of generating functions within the framework of k-analysis.

Moreover, it has been demonstrated that, by setting $k=1$, the derived generating relations reduce to the corresponding identities associated with the classical generalized Cesàro polynomials and the classical Lauricella functions [11]. Hence, the results presented herein may be viewed as a natural generalization of the k-special functions.

An additional feature of the present work is that the k-Lauricella functions studied in [6] encompass several well-known multivariable hypergeometric functions as special cases. Consequently, the generating relations established in this paper can be specialized to yield corresponding bilateral generating functions for various families of special functions. In particular, by considering suitable specializations of the generalized k-Lauricella functions, one obtains bilateral generating relations involving the generalized k-Cesàro polynomials and the k-Appell functions $F_{1,k}$, $F_{2,k}$, $F_{3,k}$ and $F_{4,k}$.

Finally, the results obtained here are expected to provide useful tools for further studies in the theory of special functions, generalized hypergeometric functions, multivariable analysis, and related areas of mathematical research. Future investigations may also explore additional applications and extensions of the generating relations developed in this work.

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