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Research Article

Well-Posedness of an Elastic-Viscoplastic Contact Problem

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Abstract: We consider a mathematical model which describes the contact of an elastic-viscoplastic body with a foundation. The contact is frictionless and is modeled with normal compliance and unilateral condition. The weak formulation of the model is in the form of a system which couples a variational inequality for the displacement field with an implicit nonlinear equation for the stress field. We prove the unique solvability of the problem by using a fixed point argument for history-dependent operators. Next, we provide necessary and sufficient conditions which guarantee the convergence to the solution of this elastic-viscoplastic contact model. We use these results in order to obtain a continuous dependence result and to introduce two well-posedness concepts for the problem. Finally, we consider a one-dimensional example which allows us to compare the two well-posedness concepts employed.

Keywords: Elastic-viscoplastic material; frictionless contact; normal compliance; unilateral constraint; variational inequality; fixed point; convergence criterion; well-posedness concepts.

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1 Introduction

Contact problems involving deformable bodies abound in industry and everyday life. Because of the importance of contact processes in structural and mechanical systems, a considerable effort has been put into their modeling and analysis, and the literature in the field is extensive (see, for instance, [1, 5, 6, 18, 21, 22]). The numerical analysis of the contact problems, including convergence results and error estimates for discrete schemes as well as various numerical simulations, can be found in [7–9, 11, 12, 17].

In most of the references above the material's behaviour was described with an elastic or viscoelastic constitutive law. This has the advantage that the stress field can be easily eliminated and the model leads to a variational or hemivariational inequality in which the unknown is the displacement or the

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velocity field. In contrast, in the current paper we assume that the material is elastic-viscoplastic and we model its behaviour with a constitutive law of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\dot{\boldsymbol{\varepsilon}} + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}). \tag{1.1}$$

Here and below in this paper $\boldsymbol{\sigma}$ denotes the stress tensor, $\boldsymbol{\varepsilon}$ is the linearized strain field, $\mathcal{E}$ is the elasticity tensor and $\mathcal{G}$ is a given viscoplastic function. Moreover, the dot above a variable represents the derivative with respect to time. Examples and mechanical interpretations of elastic-viscoplastic laws of the form (1.1) can be found in [2, 4, 8], for instance. Such constitutive laws have been used in order to model the behavior of metals, rocks, rubbers and polymers. Note that the function $\mathcal{G}$ in (1.1) involves a full couple of strain and stress. This gives rise to some mathematical difficulties since, using the above constitutive law in the structure of any boundary value problem leads to variational formulations in the form of a system in which the unknowns are a full couple of stress and displacement.

The aim of this paper is three fold. The first one is to study a nonstandard contact model for materials of the form (1.1). The contact is frictionless and is described with normal compliance and unilateral constraint. In a variational formulation, the model we consider leads to a system which combines a history-variational inequality for the displacement field with an implicit nonlinear equation for the stress field. We show that the system has a fixed point structure and, therefore, we prove its unique solvability by using a fixed point argument for history-dependent operators.

Our second aim is to provide a convergence criterion for the solution of the system. Such convergence criteria have been recently obtained in the literature, in the study of variational and hemivariational inequalities, see [23] and [24], for instance. The novelty is that here we deal with a system and therefore, providing a convergence criterion requires to use a method adapted to the complex structure of the problem, different to that used in [23, 24].

Finally, our third aim in this paper is to study the well-posedness of the above system. Here, besides the fact that we deal with a variational problem with a nonstandard structure, the novelty arises from the fact that we introduce two different well-posedness concepts, compare them, and provide the corresponding well-posedness results. This allows us to prove the continuous dependence of the solution with respect to the data. Well-posedness concepts for nonlinear problems represent an important topic in Functional Analysis. This has known a significant development in the last decades. Originating in the papers of Tykhonov [25] and Levitin-Polyak [13] (where the well-posedness of minimization problems was considered), well-posedness concepts have been extended to a large number of problems, including nonlinear equations, inequality problems, inclusions, fixed point problems, and optimal control problems. In particular, the well-posedness of variational inequalities was studied for the first time in [15, 16]. Comprehensive references in the field are [3, 10, 14, 26] and, more recently, [19].

The rest of the manuscript is structured as follows. In Section 2 we introduce the mathematical model of contact and present some notation and preliminary results. Then, we focus on the variational formulation of the problem. In Section 3 we prove the unique weak solvability of the model. To this end, we use a fixed point argument for history-dependent operators. Then, in Section 4 we state and prove a convergence criterion. It provides necessary and sufficient conditions which guarantee the convergence of a sequence of functions to the solution of the contact problem. We then apply this criterion to deduce a continuous dependence result. We proceed with Section 5, in which we introduce two well-posedness concepts in the study of our contact model and provide the corresponding well-posedness results. Moreover, we consider a one-dimensional example that we use in order to compare the well-posedness concepts previously introduced.

2 The Contact Model

This preliminary section is structured in several parts, as follows.

Problem statement. Consider an elastic-viscoplastic body which occupies, in its reference configuration, the domain $\Omega \subset \mathbb{R}^d$ ($d \in \{1,2,3\}$) with regular boundary Γ. The outward unit normal at Γ will be denoted by $\boldsymbol{\nu}$ and we use the indices ν and τ to denote the normal and tangential components of vectors and tensors, respectively. The body is held fixed on a part Γ_1 of its boundary, is acted by a time-dependent surface traction on Γ_2 and is in frictionless contact with an obstacle on Γ_3, the so-called foundation. We assume that Γ_1, Γ_2 and Γ_3 are given measurable parts of Γ such that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ and $\Gamma_i \cap \Gamma_j = \emptyset$, for $i,\ j = 1,\ 2,\ 3,\ i \neq j$. Moreover, we assume that the $d-1$ measure of Γ_1, denoted by $meas(\Gamma_1)$, is strictly positive. The body is submitted to the action of a body force, the contact is frictionless, the process is quasistatic and the time interval of interest is $[0,T]$, with $T > 0$. In addition, $\boldsymbol{x}$ stands for a typical point in $\Omega \cup \Gamma$ and, for simplicity, we sometimes skip the dependence of various functions on the spatial variable $\boldsymbol{x}$, that is, we write $\boldsymbol{u}(t)$ instead of $\boldsymbol{u}(\boldsymbol{x},t)$, for instance. Finally, we denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$. Then, the mathematical model we associate to this physical setting is the following.

Problem $\mathcal{P}$. *Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \qquad \text{in } \Omega, \tag{2.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2, \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g, \quad \sigma_\nu(t) + p(u_\nu(t)) \le 0 \\ (u_\nu(t) - g)(\sigma_\nu(t) + p(u_\nu(t)) = 0, \end{array}\right\} \qquad \text{on } \Gamma_3, \tag{2.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \qquad \text{on } \Gamma_3 \tag{2.6}$$

for all $t \in [0,T]$ and, in addition,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \qquad \text{in } \Omega. \tag{2.7}$$

We now provide a short description of the equations and conditions (2.1)–(2.7) and we send the reader to [22] for additional details and mechanical interpretations. First, equation (2.1) represents the elastic-viscoplastic constitutive law, (1.1). Equation (2.2) is the equilibrium equation, in which $\boldsymbol{f}_0$ denotes the density of body forces. We use this equation since we assume that the contact process is quasistatic. Next, conditions (2.3) and (2.4) are the displacement and traction boundary conditions, in which $\boldsymbol{f}_2$ represents the density of surface tractions. Conditions (2.5) are the contact conditions in which g is a given bound and p represents a normal compliance function which will be described below. These conditions model the contact with a rigid foundation covered by a layer of thickness g, made of an elastic material. Condition (2.6) is the frictionless condition and, finally, (2.7) represent the initial conditions, in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the given initial displacement and initial stress, respectively.

Function spaces. We now introduce the function spaces we need in the variational analysis of Problem $\mathcal{P}$. First, we use " $\cdot$ " and "$\|\cdot\|$" to represent the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. Moreover, we use the standard notation for the Lebesgue and

Sobolev spaces associated to Ω and Γ. Typical examples are the spaces $L^2(\Omega)^d$, $L^2(\Gamma)^d$, $H^1(\Omega)^d$, equipped with their canonical Hilbertian structure. For an element $\boldsymbol{v} \in H^1(\Omega)^d$ we still write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v} \in L^2(\Gamma)^d$ and $v_\nu \in L^2(\Gamma)$ for the normal trace on the boundary, i.e., $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$. Moreover, $\boldsymbol{\varepsilon}(\boldsymbol{v})$ will denote the symmetric part of the spatial gradient of $\boldsymbol{v}$, i.e.,

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T\big). \tag{2.8}$$

Next, we introduce the spaces V and Q, defined as follows:

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega) \ : \ v_i = 0 \ \text{ on } \ \Gamma_1 \quad \forall\, i = \overline{1,d}\,\}, \tag{2.9}$$

$$Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \quad \forall\, i,\, j = \overline{1,d}\,\}. \tag{2.10}$$

The spaces V and Q are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx. \tag{2.11}$$

The associated norms on these spaces will be denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Recall that the completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality. Note also that, by the definition of the inner product in the spaces V and Q, we have

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \qquad \forall\,\boldsymbol{v} \in V \tag{2.12}$$

and, using the Sobolev theorem, we deduce that

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^d} \le c_0\,\|\boldsymbol{v}\|_V \qquad \forall\,\boldsymbol{v} \in V. \tag{2.13}$$

Here c_0 is a positive constant which depends on Ω and Γ_1. We also denote by $(V^*, \|\cdot\|_{V^*})$ the dual of the space V, $\langle\cdot,\cdot\rangle$ will represent the duality pairing between V and V^* and $\boldsymbol{0}_V$, $\boldsymbol{0}_{V^*}$ will denote the zero elements of the spaces V and V^*, respectively.

History-dependent operators. For any normed space $(Z, \|\cdot\|_Z)$ we denote by $C([0,T];Z)$ the space of continuous functions defined on $[0,T]$ with values in Z. It is well known that if Z is a Banach space, then $C([0,T];Z)$ is a Banach space, equipped with the norm

$$\|\boldsymbol{v}\|_{C([0,T];Z)} = \max_{t\in[0,T]} \|\boldsymbol{v}(t)\|_Z \qquad \forall\,\boldsymbol{v} \in C([0,T];Z).$$

Moreover, if $U \subset Z$, we denote by $C([0,T];U)$ the set of functions in $C([0,T];Z)$ with values in U.

Let now $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. We recall that an operator $\Theta : C([0,T],Z) \to C([0,T],Y)$ is said to be a history-dependent operator if there exists $L > 0$ such that

$$\|\Theta v(t) - \Theta w(t)\|_Y \le L \int_0^t \|v(s) - w(s)\|_Z\,ds \qquad \forall\, v,\, w \in C([0,T],Z),\ t \in [0,T].$$

Here and below we use the shorthand notation $\Theta v(t)$ to represent the value of the function Θu at the point t, i.e., $\Theta u(t) = (\Theta u)(t)$, for all $v \in C([0,T];Z)$ and $t \in [0,T]$. Introduced in [20], history-dependent operators have a number of useful properties, that can be found in the books [21, 22]. In particular, we shall need the following fixed point result.

Theorem 2.1. *Let $(Z, \|\cdot\|_Z)$ be a Banach space and let $\Theta : C([0,T],Z) \to C([0,T],Z)$ be a history-dependent operator. Then, Θ has a unique fixed point, that is, there exists a unique function $\eta^* \in C([0,T],Z)$ such that $\Theta\eta^* = \eta^*$.*

Remark 2.1. *A proof of Theorem 2.1 can be found in [22, p.41]. There, it is shown that there exists a positive integer m such that a power Λ^m of Λ is a contraction on the space $C([0,T],Z)$. Then, the proof is based on the Banach fixed point principle.*

Variational formulation. In the study of Problem $\mathcal{P}$ we assume that the elasticity operator $\mathcal{E}$ and the viscoplastic function $\mathcal{G}$ satisfy the following assumptions.

$$\left\{\begin{array}{l} \mathcal{E}=(\mathcal{E}_{ijkl})\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d \text{ is such that:}\\[1ex] \text{(a) } \mathcal{E}_{ijkl}=\mathcal{E}_{klij}=\mathcal{E}_{jikl}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d.\\[1ex] \text{(b) There exists } m_{\mathcal{E}}>0 \text{ such that}\\ \qquad \mathcal{E}\boldsymbol{\tau}\cdot\boldsymbol{\tau}\ge m_{\mathcal{E}}\|\boldsymbol{\tau}\|^2 \text{ for all } \boldsymbol{\tau}\in\mathbb{S}^d,\ \text{a.e. in } \Omega. \end{array}\right. \tag{2.14}$$

$$\left\{\begin{array}{l} \mathcal{G}\colon \Omega\times\mathbb{S}^d\times\mathbb{S}^d\to\mathbb{S}^d \text{ is such that:}\\[1ex] \text{(a) There exists } L_{\mathcal{G}}>0 \text{ such that}\\ \qquad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1)-\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\|\\ \qquad\quad \le L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|)\\ \qquad\qquad \text{for all } \boldsymbol{\sigma}_1,\,\boldsymbol{\sigma}_2,\,\boldsymbol{\varepsilon}_1,\,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\[1ex] \text{(b) } \mathcal{G}(\cdot,\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable in } \Omega \text{ for all } \boldsymbol{\sigma},\,\boldsymbol{\varepsilon}\in\mathbb{S}^d.\\[1ex] \text{(c) } \mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0})=\boldsymbol{0} \text{ for a.e. } \boldsymbol{x}\in\Omega. \end{array}\right. \tag{2.15}$$

The densities of the body forces and tractions satisfy the conditions

$$\boldsymbol{f}_0\in C([0,T];L^2(\Omega)^d),\qquad \boldsymbol{f}_2\in C([0,T];L^2(\Gamma_2)^d), \tag{2.16}$$

and the normal compliance function is as follows.

$$\left\{\begin{array}{l} p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+ \text{ is such that:}\\[1ex] \text{(a) There exists } L_p>0 \text{ such that}\\ \qquad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\le L_p\,|r_1-r_2|\\ \qquad\quad \forall\, r_1,\, r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\[1ex] \text{(b) } (p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2))(r_1-r_2)\ge 0\\ \qquad\quad \forall\, r_1,\, r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\[1ex] \text{(c) The mapping } \boldsymbol{x}\mapsto p(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\\ \qquad \text{for any } r\in\mathbb{R}.\\[1ex] \text{(d) } p(\boldsymbol{x},r)=0 \text{ for all } r\le 0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3. \end{array}\right. \tag{2.17}$$

We recall that

$$g>0 \tag{2.18}$$

and, finally, we suppose that the initial data have the regularity

$$\boldsymbol{u}_0\in V,\qquad \boldsymbol{\sigma}_0\in Q. \tag{2.19}$$

We now move to the variational formulation of Problem $\mathcal{P}$. To this end, we consider the set of admissible displacement fields defined by

$$K=\{\,\boldsymbol{v}\in V\,:\,\boldsymbol{v}_\nu\le g\ \text{ a.e. on } \Gamma_3\,\}, \tag{2.20}$$

and we note that

$$K \text{ is a nonempty closed convex subset of } V. \tag{2.21}$$

Then, following a standard approach based on integration by parts and relations (2.2)–(2.6) it follows that for any $t \in [0, T]$, the following inequality holds:

$$\begin{aligned} u(t) \in K, \quad & \int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\, dx + \int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\, d\Gamma \\ & \geq \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma \quad \forall v \in K. \end{aligned} \tag{2.22}$$

Next, we integrate the equation (2.1) with the initial conditions (2.7) to obtain that

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall t \in [0, T]. \tag{2.23}$$

We now use assumptions (2.17) and (2.16) to define the operator $P : V \to V^*$ and the function $\boldsymbol{f} : [0, T] \to V^*$ by equalities

$$\langle P\boldsymbol{u}, \boldsymbol{v} \rangle = \int_{\Gamma_3} p(\boldsymbol{u}_\nu)\boldsymbol{v}_\nu\, d\Gamma \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{2.24}$$

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, d\Gamma \quad \forall t \in [0, T], \quad \forall \boldsymbol{v} \in V. \tag{2.25}$$

Note that assumption (2.17) combined with inequality (2.13) shows that $P : V \to V^*$ is a monotone Lipschitz continuous operator, that is

$$\begin{cases} \text{(a)} \quad \langle P\boldsymbol{u} - P\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle \geq 0 \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \\ \text{(b)} \quad \|P\boldsymbol{u} - P\boldsymbol{v}\|_{V^*} \leq c_0^2 L_p \|\boldsymbol{u} - \boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \end{cases} \tag{2.26}$$

Moreover, the regularity (2.16) implies that

$$\boldsymbol{f} \in C([0, T]; V^*). \tag{2.27}$$

Finally, we combine inequality (2.22) with equation (2.23) and use notation (2.24), (2.25) to obtain the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. *Find a displacement field* $\boldsymbol{u} : [0, T] \to K$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \to Q$ *such that*

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + \langle P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle \quad \forall \boldsymbol{v} \in K, \tag{2.28}$$

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{2.29}$$

for all $t \in [0, T]$.

The unique solvability of Problem $\mathcal{P}_V$ will be provided in the next section, by using a fixed point argument. Here, we restrict ourselves to note that a pair of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (2.29) and (2.28) is called a *weak solution* to Problem $\mathcal{P}$.

3 Weak Solvability

To provide the existence of a unique solution to Problem $\mathcal{P}_V$ we need additional notation and preliminaries. To this end, everywhere in this section we assume that (2.14)–(2.19) hold, even if we do not mention it explicitly. We start by considering the operator $\boldsymbol{\varepsilon} : V \to Q$ defined by (2.8) and denote by $\boldsymbol{\varepsilon}^* : Q \to V^*$ its adjoint. Then,

$$\langle \boldsymbol{\varepsilon}^*(\boldsymbol{\eta}), \boldsymbol{v}\rangle = (\boldsymbol{\eta}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\, \boldsymbol{\eta} \in Q,\, \boldsymbol{v} \in V \tag{3.1}$$

and, therefore, (2.12) implies that

$$\|\boldsymbol{\varepsilon}^*(\boldsymbol{\eta})\|_{V^*} \le \|\boldsymbol{\eta}\|_Q \quad \forall\, \boldsymbol{\eta} \in Q. \tag{3.2}$$

Moreover, assumption (2.14) yields

$$\|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \le M_{\mathcal{E}}\|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V \tag{3.3}$$

with some constant $M_{\mathcal{E}} > 0$. Inequalities (3.2) and (3.3) will be used in various places in the rest of the manuscript.

Next, we consider the operator $A : V \to V^*$ defined by quality

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \langle P\boldsymbol{u}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V \tag{3.4}$$

and note that, using (2.14), (3.3) and (2.26), it follows that A is a strongly monotone Lipschitz continuous operator, that is

$$\left\{\begin{array}{lll} \text{(a)} & \langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle \ge m_{\mathcal{E}}\|\boldsymbol{u} - \boldsymbol{v}\|_V^2 & \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \\[2mm] \text{(b)} & \|A\boldsymbol{u} - A\boldsymbol{v}\|_{V^*} \le (M_{\mathcal{E}} + c_0^2 L_p)\|\boldsymbol{u} - \boldsymbol{v}\|_V & \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \end{array}\right. \tag{3.5}$$

Moreover, using the properties (2.21) and (3.4) of the set K and the operator A, a standard argument on elliptic variational inequalities guarantees that for any $\boldsymbol{\omega} \in V^*$ there exists a unique element $\boldsymbol{z}$ such that

$$\boldsymbol{z} \in K, \quad \langle A\boldsymbol{z}, \boldsymbol{v} - \boldsymbol{z}\rangle \ge \langle \boldsymbol{\omega}, \boldsymbol{v} - \boldsymbol{z}\rangle \quad \forall\, \boldsymbol{v} \in K.$$

We denote by $\mathcal{F} : V^* \to K$ the solution map of this inequality, that is,

$$\boldsymbol{z} = \mathcal{F}\boldsymbol{\omega} \iff \boldsymbol{z} \in K, \quad \langle A\boldsymbol{z}, \boldsymbol{v} - \boldsymbol{z}\rangle \ge \langle \boldsymbol{\omega}, \boldsymbol{v} - \boldsymbol{z}\rangle \quad \forall\, \boldsymbol{v} \in K, \tag{3.6}$$

for any $\boldsymbol{\omega} \in V^*$. Moreover, it is easy to see that $\mathcal{F}$ is a Lipschitz continuous operator with constant $\frac{1}{m_{\mathcal{E}}}$, i.e.,

$$\|\mathcal{F}\boldsymbol{\omega}_1 - \mathcal{F}\boldsymbol{\omega}_2\|_V \le \frac{1}{m_{\mathcal{E}}}\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{V^*} \quad \forall\, \boldsymbol{\omega}_1,\, \boldsymbol{\omega}_2 \in V^*. \tag{3.7}$$

To proceed, we consider the product space $X = V \times Q$ endowed with the norm

$$\|\boldsymbol{\theta}\|_X = \|\boldsymbol{u}\|_V + \|\boldsymbol{\sigma}\|_Q \quad \forall\, \boldsymbol{\theta} = (\boldsymbol{u}, \boldsymbol{\sigma}) \in X, \tag{3.8}$$

together with the operators $\Lambda_0 : C([0,T];X) \to C([0,T];Q)$, $\Lambda_1 : C([0,T];X) \to C([0,T];V)$, $\Lambda_2 : C([0,T];X) \to C([0,T];Q)$ and $\Lambda : C([0,T];X) \to C([0,T];X)$ defined as follows:

$$\Lambda_0\boldsymbol{\theta}(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{3.9}$$

$$\Lambda_1\boldsymbol{\theta}(t) = \mathcal{F}(\boldsymbol{f}(t) - \boldsymbol{\varepsilon}^*\Lambda_0\boldsymbol{\theta}(t)), \tag{3.10}$$

$$\Lambda_2\boldsymbol{\theta}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\Lambda_1\boldsymbol{\theta}(t)) + \Lambda_0\boldsymbol{\theta}(t), \tag{3.11}$$

$$\Lambda\boldsymbol{\theta}(t) = (\Lambda_1\boldsymbol{\theta}(t), \Lambda_2\boldsymbol{\theta}(t)), \tag{3.12}$$

for any $\boldsymbol{\theta} = (\boldsymbol{u}, \boldsymbol{\sigma}) \in X$, $t \in [0, T]$. Note that the assumptions on the data combined with the regularity (2.27) and inequality (3.7) guarantee that the operators Λ_0, Λ_1 and Λ_2 and Λ are well-defined.

We now state our main result in this section.

Theorem 3.1. *Assume* (2.14)–(2.19)*. Then, there exists a unique solution to Problem $\mathcal{P}_V$. Moreover, the solution satisfies $\boldsymbol{u} \in C([0, T]; K)$ and $\boldsymbol{\sigma} \in C([0, T]; Q)$.*

The proof of Theorem 3.1 is carried out in several steps that we present in what follows.

Lemma 3.1. *Let $\boldsymbol{\theta} = (\boldsymbol{u}, \boldsymbol{\sigma}) \in C([0, T]; X)$. Then, the couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution to Problem $\mathcal{P}_V$ if and only if $\Lambda\boldsymbol{\theta} = \boldsymbol{\theta}$.*

Proof. Let $t \in [0, T]$. We use definitions (3.9), (3.4) to see that the following equivalence holds:

$$\left.\begin{array}{l} (\boldsymbol{u}(t), \boldsymbol{\sigma}(t)) \text{ satisfy (2.28) and (2.29)} \iff \\[2ex] \left\{\begin{array}{ll} \text{(a)} & \langle A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + (\Lambda_0\boldsymbol{\theta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ & \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle \quad \forall\, \boldsymbol{v} \in K, \\[2ex] \text{(b)} & \boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \Lambda_0\boldsymbol{\theta}(t). \end{array}\right. \end{array}\right\} \tag{3.13}$$

On the other hand, using equality (3.1) and definition (3.6), it follows that

$$\begin{array}{l} \boldsymbol{u}(t) \text{ satisfies the inequality (3.13) (a)} \iff \\[1ex] \langle A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle \geq \langle \boldsymbol{f}(t) - \boldsymbol{\varepsilon}^*\Lambda_0\boldsymbol{\theta}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle \quad \forall\, \boldsymbol{v} \in K \iff \\[1ex] \boldsymbol{u}(t) = \mathcal{F}(\boldsymbol{f}(t) - \boldsymbol{\varepsilon}^*\Lambda_0\boldsymbol{\theta}(t)). \end{array}$$

Therefore, definition (3.10) shows that

$$\boldsymbol{u}(t) \text{ satisfies inequality (3.13) (a)} \iff \boldsymbol{u}(t) = \Lambda_1\boldsymbol{\theta}(t). \tag{3.14}$$

Using now (3.14) and (3.11) it follows that

$$(\boldsymbol{u}(t), \boldsymbol{\sigma}(t)) \text{ satisfy (3.13) (a), (b)} \iff \boldsymbol{u}(t) = \Lambda_1\boldsymbol{\theta}(t), \quad \boldsymbol{\sigma}(t) = \Lambda_2\boldsymbol{\theta}(t). \tag{3.15}$$

Then, combining the equivalences (3.13) and (3.15) we deduce that

$$(\boldsymbol{u}(t), \boldsymbol{\sigma}(t)) \text{ satisfy (2.28) and (2.29)} \iff \boldsymbol{u}(t) = \Lambda_1\boldsymbol{\theta}(t), \quad \boldsymbol{\sigma}(t) = \Lambda_2\boldsymbol{\theta}(t). \tag{3.16}$$

Finally, we use (3.16) and the definition (3.12) of the operator Λ to conclude the proof. □

We now proceed with the following result.

Lemma 3.2. *The operator $\Lambda : C([0, T]; X) \to C([0, T]; X)$ is a history-dependent operator.*

Proof. Let $\boldsymbol{\theta} = (\boldsymbol{u}, \boldsymbol{\sigma}) \in C([0, T]; X)$, $\boldsymbol{\eta} = (\boldsymbol{v}, \boldsymbol{\tau}) \in C([0, T]; X)$ and let $t \in [0, T]$. We use definition (3.12) and equality (3.8) to see that

$$\|\Lambda\boldsymbol{\theta}(t) - \Lambda\boldsymbol{\eta}(t)\|_X = \|\Lambda_1\boldsymbol{\theta}(t) - \Lambda_1\boldsymbol{\eta}(t)\|_V + \|\Lambda_2\boldsymbol{\theta}(t) - \Lambda_2\boldsymbol{\eta}(t)\|_Q. \tag{3.17}$$

Next, the definitions (3.10) and inequalities (3.7), (3.2) show that

$$\|\Lambda_1\boldsymbol{\theta}(t) - \Lambda_1\boldsymbol{\eta}(t)\|_V \le C\|\Lambda_0\boldsymbol{\theta}(t) - \Lambda_0\boldsymbol{\eta}(t)\|_Q \tag{3.18}$$

where, here and below, C represents a generic positive constant which does not depend on t and whose value will change from place to place. Moreover, the definition (3.11) and inequalities (3.3), (3.18) yields

$$\|\Lambda_2\boldsymbol{\theta}(t) - \Lambda_2\boldsymbol{\eta}(t)\|_Q \le C\|\Lambda_0\boldsymbol{\theta}(t) - \Lambda_0\boldsymbol{\eta}(t)\|_Q. \tag{3.19}$$

Note also that the definition (3.9) of the operator Λ_0 and assumption (2.15) imply that

$$\|\Lambda_0\boldsymbol{\theta}(t) - \Lambda_0\boldsymbol{\eta}(t)\|_Q \le C\int_0^t (\|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|_V + \|\boldsymbol{\sigma}(s) - \boldsymbol{\tau}(s)\|_Q)ds,$$

and, using (3.8), we find that

$$\|\Lambda_0\boldsymbol{\theta}(t) - \Lambda_0\boldsymbol{\eta}(t)\|_Q \le C\int_0^t \|\boldsymbol{\theta}(s) - \boldsymbol{\eta}(s)\|_X ds. \tag{3.20}$$

We now combine (3.17)–(3.20) to see that

$$\|\Lambda\boldsymbol{\theta}(t) - \Lambda\boldsymbol{\eta}(t)\|_X \le C\int_0^t \|\boldsymbol{\theta}(s) - \boldsymbol{\eta}(s)\|_X ds, \tag{3.21}$$

which concludes the proof. □

Remark 3.1. *It follows from the proof of Lemma 3.2 that, besides the operator Λ, the operators Λ_0, Λ_1 and Λ_2 are history-dependent operators, too. This implies that they are Lipschitz continuous. We shall use these properties in Section 4 below.*

We are now in a position to provide the proof of Theorem 3.1.

Proof. Theorem 3.1 is a direct consequence of Lemmas 3.1, 3.2 and Theorem 2.1. □

We end this section with the remark that the fixed point problem of finding a function $\boldsymbol{\theta} \in C([0,T];X)$ such that $\Lambda\boldsymbol{\theta} = \boldsymbol{\theta}$ represents an alternative to Problem $\mathcal{P}_V$. Indeed, in various estimates we present below it will be more convenient to use this fixed point problem instead of the system (2.28), (2.29).

4 A Convergence Criterion

In this section we provide necessary and sufficient conditions which guarantee the uniform convergence of an arbitrary sequence $\{\boldsymbol{\theta}_n\} = \{(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)\} \subset C([0,T];X)$ to the solution $\boldsymbol{\theta} = (\boldsymbol{u}, \boldsymbol{\sigma})$ of Problem $\mathcal{P}_V$. To this end we use the symbol "$\to$" to indicate the convergence in various normed spaces that will be specified, except in the case when these convergences take place in $\mathbb{R}$. All the limits are considered as $n \to \infty$, even if we do not mention it explicitly. For a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ which converges to zero we use the short hand notation $0 \le \varepsilon_n \to 0$ and we denote by $d(\boldsymbol{v}, K)$ the distance between the element $\boldsymbol{v} \in V$ and the set K, in the Hilbertian structure of the space V. Finally, we recall that below in this section C will represent a generic positive constant which does not depend on $n \in \mathbb{N}$ and $t \in [0,T]$ and whose value will change from place to place.

Assume that (2.14)–(2.19) hold and, given a sequence $\{\boldsymbol{\theta}_n\} = \{(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)\} \subset C([0,T];X)$, we consider the following statements.

(S_1) $\quad \boldsymbol{u}_n \to \boldsymbol{u}$ in $C([0,T];V)$ and $\boldsymbol{\sigma}_n \to \boldsymbol{\sigma}$ in $C([0,T];Q)$.

(S_2) $\quad$ There exists $0 \le \varepsilon_n \to 0$ such that

$$\left\{\begin{array}{l} \text{(a)} \quad d(\boldsymbol{u}_n(t),K) \le \varepsilon_n, \\[2mm] \text{(b)} \quad (\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + \langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \\ \qquad\quad +\varepsilon_n(1 + \|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V) \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \quad \forall \boldsymbol{v} \in K, \\[2mm] \text{(c)} \quad \left\|\boldsymbol{\sigma}_n(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \displaystyle\int_0^t \mathcal{G}(\boldsymbol{\sigma}_n(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(s)))\, ds - \boldsymbol{\sigma}_0 + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\right\|_Q \le \varepsilon_n, \\[2mm] \text{for all } n \in \mathbb{N} \text{ and } t \in [0,T]. \end{array}\right. \tag{4.1}$$

(S_3) $\quad \boldsymbol{u}_n - \Lambda_1\boldsymbol{\theta}_n \to \mathbf{0}_V$ in $C([0,T];V)$ and $\boldsymbol{\sigma}_n - \Lambda_2\boldsymbol{\theta}_n \to \mathbf{0}_Q$ in $C([0,T];Q)$.

Our result in this section is the following.

Theorem 4.1. *Assume (2.14)–(2.19). Then, the statements (S_1), (S_2) and (S_3) are equivalent.*

The proof of Theorem 4.1 is based on the following preliminary results.

Lemma 4.1. *Assume (2.14)–(2.19) and let $\{\boldsymbol{\theta}_n\} = \{(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)\} \subset C([0,T];X)$ be a sequence which satisfies conditions (4.1)(b),(c). Then, there exists a constant $M > 0$ which does not depend on n and t such that*

$$\|\boldsymbol{\theta}_n(t)\|_X \le M, \qquad \|\Lambda_0\boldsymbol{\theta}_n(t)\|_Q \le M, \qquad \forall\, n \in \mathbb{N},\ t \in [0,T]. \tag{4.2}$$

Proof. Let $n \in \mathbb{N}$ and $t \in (0,T]$. We take $\boldsymbol{v} = \mathbf{0}_V$ in (4.1)(b) and use the equality $P\mathbf{0}_V = \mathbf{0}_{V^*}$ combined with the monotonicity of P to see that

$$(\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q \le \varepsilon_n(1 + \|\boldsymbol{u}_n(t)\|_V) + \langle \boldsymbol{f}(t), \boldsymbol{u}_n(t)\rangle.$$

Therefore,

$$\begin{aligned} &(\boldsymbol{\sigma}_n(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) + \Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q \\ &\quad \le \varepsilon_n(1 + \|\boldsymbol{u}_n(t)\|_V) + \langle \boldsymbol{f}(t), \boldsymbol{u}_n(t)\rangle, \end{aligned}$$

which implies that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q \le \|\boldsymbol{\sigma}_n(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \Lambda_0\boldsymbol{\theta}_n(t)\|_Q \|\boldsymbol{u}_n(t)\|_V \tag{4.3}$$

$$+\|\Lambda_0\boldsymbol{\theta}_n(t)\|_Q\|\boldsymbol{u}_n(t)\|_V + \varepsilon_n(1 + \|\boldsymbol{u}_n(t)\|_V) + \|\boldsymbol{f}(t)\|_{V^*}\|\boldsymbol{u}_n(t)\|_V. \tag{4.4}$$

We now use assumption (S_2)(c) and definition (3.9) to see that

$$\|\boldsymbol{\sigma}_n(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \Lambda_0\boldsymbol{\theta}_n(t)\|_Q \le \varepsilon_n. \tag{4.5}$$

Next, we combine inequalities (4.3) and (4.5), use assumptions (2.14), the boundedness of the sequence $\{\varepsilon_n\}$ and the regularity (2.27) to see that there exists a constant $C > 0$ such that

$$\|\boldsymbol{u}_n(t)\|_V^2 \le C(1 + \|\Lambda_0\boldsymbol{\theta}_n(t)\|_Q)\|\boldsymbol{u}_n(t)\|_V + C. \tag{4.6}$$

Next, we use the elementary inequality

$$x^2 \le ax + b \quad \Longrightarrow \quad x \le a + \sqrt{b} \qquad \forall\, x,\ a,\ b \in [0, \infty) \tag{4.7}$$

to find that

$$\|\boldsymbol{u}_n(t)\|_V \le C\|\Lambda_0\boldsymbol{\theta}_n(t)\|_Q + C. \tag{4.8}$$

On the other hand, inequalities (4.5) and (4.8) yield

$$\|\boldsymbol{\sigma}_n(t)\|_Q \le C\|\Lambda_0\boldsymbol{\theta}_n(t)\|_Q + C. \tag{4.9}$$

We now add inequalities (4.8), (4.9), use (3.8), the definition (3.9) and the properties (2.15) of the function $\mathcal{G}$ to see that

$$\|\boldsymbol{\theta}_n(t)\|_X \le C\int_0^t \|\boldsymbol{\theta}_n(s)\|_Q ds + C. \tag{4.10}$$

The inequalities in (4.2) follow now from (4.10), by using the Gronwall argument and inequality (3.20). $\square$

We now provide the proof of Theorem 4.1.

Proof. The proof is carried in three steps, as follows.

Step i) We prove the implication $(S_1) \Longrightarrow (S_2)$. Assume (S_1). Let $n \in \mathbb{N}$, $t \in [0,T]$ and $\boldsymbol{v} \in K$. Then, since $\boldsymbol{u}(t) \in K$ it follows that

$$d(\boldsymbol{u}_n(t), K) \le \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V \le \delta_n \tag{4.11}$$

where, here and below,

$$\delta_n = \|\boldsymbol{u}_n - \boldsymbol{u}\|_{C([0,T];V)}. \tag{4.12}$$

On the other hand, (2.28), (2.29) and the definitions (3.4), (3.9) show that

$$\langle A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + (\Lambda_0\boldsymbol{\theta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle$$

and, therefore,

$$\begin{aligned}
&\langle A\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle + \langle A\boldsymbol{u}_n(t) - A\boldsymbol{u}(t), \boldsymbol{u}_n(t) - \boldsymbol{v}\rangle + \langle A\boldsymbol{u}(t), \boldsymbol{u}_n(t) - \boldsymbol{u}(t)\rangle\\
&+(\Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + (\Lambda_0\boldsymbol{\theta}_n(t) - \Lambda_0\boldsymbol{\theta}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\\
&-(\Lambda_0\boldsymbol{\theta}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + \langle \boldsymbol{f}(t), \boldsymbol{u}(t) - \boldsymbol{u}_n(t)\rangle \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle.
\end{aligned}$$

Denote by L_A and L_0 the Lipschitz constants of the operators A and Λ_0, respectively, see (3.5)(b) and Remark 3.1. Then, after some algebra, the previous inequality combined with notation (4.12) yields

$$\begin{aligned}
&\langle A\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle + L_A\delta_n\|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V + \delta_n\|A\boldsymbol{u}(t)\|_{V^*}\\
&+(\Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + L_0\|\boldsymbol{\theta}_n - \boldsymbol{\theta}\|_{C([0,T];X)}\|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V\\
&+\delta_n\|\Lambda_0\boldsymbol{\theta}(t)\|_Q + \delta_n\|\boldsymbol{f}(t)\|_{V^*} \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle.
\end{aligned} \tag{4.13}$$

Let $D > 0$ be a constant which does not depend on n and t such that

$$\|A\boldsymbol{u}(t)\|_{V^*} + \|\Lambda_0\boldsymbol{\theta}(t)\|_Q + \|\boldsymbol{f}(t)\|_{V^*} \le D \tag{4.14}$$

and let

$$\zeta_n = \|\boldsymbol{\theta}_n - \boldsymbol{\theta}\|_{C([0,T];X)}. \tag{4.15}$$

We combine (4.13)–(4.15) to find that

$$\begin{aligned}&\langle A\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle + (\Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q\\ &\quad +(L_A\delta_n + L_0\zeta_n)\|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V + D\delta_n \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle\end{aligned}$$

and, using notation (3.4), we find that

$$\begin{aligned}&(\overline{\boldsymbol{\sigma}}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + \langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle\\ &\quad +(L_A\delta_n + L_0\zeta_n)\|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V + D\delta_n \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle\end{aligned} \tag{4.16}$$

where

$$\overline{\boldsymbol{\sigma}}_n(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) + \Lambda_0\boldsymbol{\theta}_n(t). \tag{4.17}$$

Next, we use equivalence (3.15) and definition (3.11) to see that

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \Lambda_0\boldsymbol{\theta}(t).$$

Using this equality we write

$$\begin{aligned}&\|\boldsymbol{\sigma}_n(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \Lambda_0\boldsymbol{\theta}_n(t)\|_Q \leq \|\boldsymbol{\sigma}_n(t) - \boldsymbol{\sigma}(t)\|_Q\\ &\quad +\|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t))\|_Q + \|\Lambda_0\boldsymbol{\theta}(t) - \Lambda_0\boldsymbol{\theta}_n(t)\|_Q,\end{aligned}$$

and, using assumption (2.14), the Lipschitz continuity of the operator Λ_0 and equality (3.8), we find that

$$\|\boldsymbol{\sigma}_n(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \Lambda_0\boldsymbol{\theta}_n(t)\|_Q \leq E\|\boldsymbol{\theta}_n - \boldsymbol{\theta}\|_{C([0,T];X)},$$

with $E > 0$ being a constant which does not depend on n and t. Hence, definition (4.15) implies that

$$\|\boldsymbol{\sigma}_n(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \Lambda_0\boldsymbol{\theta}_n(t)\|_Q \leq E\zeta_n \tag{4.18}$$

and, combining this inequality with definition (4.17) yields

$$\|\boldsymbol{\sigma}_n(t) - \overline{\boldsymbol{\sigma}}_n(t)\|_Q \leq E\zeta_n. \tag{4.19}$$

Finally, writing $\overline{\boldsymbol{\sigma}}_n(t) = \overline{\boldsymbol{\sigma}}_n(t) - \boldsymbol{\sigma}_n(t) + \boldsymbol{\sigma}_n(t)$ in (4.16) and using inequality (4.19) we deduce that

$$\begin{aligned}&(\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + \langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle\\ &\quad +(E\zeta_n + L_A\delta_n + L_0\zeta_n)\|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V + D\delta_n \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle.\end{aligned} \tag{4.20}$$

Define

$$\varepsilon_n = \max\Big\{\delta_n, E\zeta_n + L_A\delta_n + L_0\zeta_n, D\delta_n\Big\},$$

and note that (4.12), (4.15) and assumption (S_1) show that $0 \leq \varepsilon_n \to 0$. Moreover, inequalities (4.11), (4.20), (4.18) and definition (3.9) guarantee that condition (S_2) is satisfied.

Step ii) We prove the implication $(S_2) \implies (S_3)$. Assume (S_2). Let $n \in \mathbb{N}$ and $t \in [0,T]$. Denote

$$\boldsymbol{\tau}_n(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_n(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) + \Lambda_0\boldsymbol{\theta}_n(t), \tag{4.21}$$

and note that condition (S_2) (c) implies that

$$\|\boldsymbol{\sigma}_n(t) - \boldsymbol{\tau}_n(t)\|_Q \le \varepsilon_n. \tag{4.22}$$

On the other hand, using condition (S_2) (b) and (4.21) we find that

$$\begin{aligned}
&(\boldsymbol{\sigma}_n(t) - \boldsymbol{\tau}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q \\
&\quad + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) + \Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + \langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \\
&\qquad + \varepsilon_n(1 + \|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V) \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \quad \forall\, \boldsymbol{v} \in K.
\end{aligned}$$

Then, using inequality (4.22) combined with definition (3.4), we deduce that

$$\begin{aligned}
&\langle A\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle + (\Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q \\
&\quad + \varepsilon_n(1 + 2\|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V) \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \quad \forall\, \boldsymbol{v} \in K.
\end{aligned} \tag{4.23}$$

Let

$$\Lambda_1\boldsymbol{\theta}_n(t) = \widetilde{\boldsymbol{u}}_n(t), \quad \Lambda_2\boldsymbol{\theta}_n(t) = \widetilde{\boldsymbol{\sigma}}_n(t). \tag{4.24}$$

Note that definitions (3.10), (3.6), (3.11) and (3.1) imply that

$$\begin{aligned}
&\widetilde{\boldsymbol{u}}_n(t) \in K, \quad \langle A\widetilde{\boldsymbol{u}}_n(t), \boldsymbol{v} - \widetilde{\boldsymbol{u}}_n(t)\rangle + (\Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_n(t)))_Q \\
&\quad \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \widetilde{\boldsymbol{u}}_n(t)\rangle \quad \forall\, \boldsymbol{v} \in K,
\end{aligned} \tag{4.25}$$

$$\widetilde{\boldsymbol{\sigma}}_n(t) = \mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_n(t)) + \Lambda_0\boldsymbol{\theta}_n(t). \tag{4.26}$$

We denote $\boldsymbol{v}_n(t) = P_K\boldsymbol{u}_n(t)$ and $\boldsymbol{w}_n(t) = \boldsymbol{u}_n(t) - P_K\boldsymbol{u}_n(t)$, where $P_K : V \to K$ represents the projection operator on K. We have

$$\boldsymbol{u}_n(t) = \boldsymbol{v}_n(t) + \boldsymbol{w}_n(t), \quad \boldsymbol{v}_n(t) \in K \tag{4.27}$$

and, since $d(\boldsymbol{u}_n(t), K) = \|\boldsymbol{w}_n(t)\|_V$, condition (S_2) (a) implies that

$$\|\boldsymbol{w}_n(t)\|_X \le \varepsilon_n. \tag{4.28}$$

We now use (4.23) with $\boldsymbol{v} = \widetilde{\boldsymbol{u}}_n(t)$, then we take $\boldsymbol{v} = \boldsymbol{v}_n(t)$ in (4.25) and add the resulting inequalities to see that

$$\begin{aligned}
&\langle A\widetilde{\boldsymbol{u}}_n(t), \boldsymbol{v}_n(t) - \widetilde{\boldsymbol{u}}_n(t)\rangle + \langle A\boldsymbol{u}_n(t), \widetilde{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\rangle \\
&\quad + (\Lambda_0\boldsymbol{\theta}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}_n(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t))_Q + \varepsilon_n + 2\varepsilon_n\|\widetilde{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V \ge \langle \boldsymbol{f}(t), \boldsymbol{v}_n(t) - \boldsymbol{u}_n(t)\rangle.
\end{aligned} \tag{4.29}$$

We now write

$$\langle A\widetilde{\boldsymbol{u}}_n(t), \boldsymbol{v}_n(t) - \widetilde{\boldsymbol{u}}_n(t)\rangle = \langle A\widetilde{\boldsymbol{u}}_n(t), \boldsymbol{v}_n(t) - \boldsymbol{u}_n(t)\rangle + \langle A\widetilde{\boldsymbol{u}}_n(t), \boldsymbol{u}_n(t) - \widetilde{\boldsymbol{u}}_n(t)\rangle,$$

substitute this identity in (4.29), then use inequality (3.5) (a) and equality (4.27) to find that

$$\begin{aligned}
m_{\mathcal{E}}\|\widetilde{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V^2 &\le \big(\|A\widetilde{\boldsymbol{u}}_n(t)\|_{V^*} + \|\Lambda_0\boldsymbol{\theta}_n(t)\|_Q + \|\boldsymbol{f}(t)\|_{V^*}\big)\|\boldsymbol{w}_n(t)\|_V \\
&\quad + \varepsilon_n + 2\varepsilon_n\|\widetilde{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V.
\end{aligned} \tag{4.30}$$

On the other hand (4.24) and the Lipschitz continuity of the operator Λ_1, guaranteed by Remark 3.1, show that

$$\|\widetilde{\boldsymbol{u}}_n(t)\|_V \leq C\|\boldsymbol{\theta}_n(t)\|_V + C. \tag{4.31}$$

Next, we use the bounds (4.31), (4.2), (4.28) in (4.30) to deduce that

$$\|\widetilde{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V^2 \leq C\varepsilon_n + 2\varepsilon_n\|\widetilde{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V,$$

with some constant $C > 0$. Then, inequality (4.7) and the convergence $\varepsilon_n \to 0$ imply that

$$\widetilde{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t) \to \boldsymbol{0}_V \quad \text{in} \quad C([0,T];V). \tag{4.32}$$

To proceed, we write

$$\|\boldsymbol{\sigma}_n(t) - \widetilde{\boldsymbol{\sigma}}_n(t)\|_Q \leq \|\boldsymbol{\sigma}_n(t) - \boldsymbol{\tau}_n(t)\|_Q + \|\boldsymbol{\tau}_n(t) - \widetilde{\boldsymbol{\sigma}}_n(t)\|_Q,$$

then we use inequality (4.22) and equalities (4.21), (4.26) to see that

$$\|\boldsymbol{\sigma}_n(t) - \widetilde{\boldsymbol{\sigma}}_n(t)\|_Q \leq \varepsilon_n + \|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_n(t))\|_Q.$$

Employing now inequality (3.3) together with the convergences (4.32), $\varepsilon_n \to 0$, we deduce that

$$\widetilde{\boldsymbol{\sigma}}_n(t) - \boldsymbol{\sigma}_n(t) \to \boldsymbol{0}_Q \quad \text{in} \quad C([0,T];Q). \tag{4.33}$$

Finally, we use the definition (4.24) and the convergences (4.32), (4.33) to find that condition (S_3) is satisfied.

Step iii). We prove the implication $(S_3) \implies (S_1)$. Assume (S_3) or, equivalently,

$$\boldsymbol{\theta}_n - \Lambda\boldsymbol{\theta}_n \to \boldsymbol{0}_X \quad \text{in} \quad C([0,T];X). \tag{4.34}$$

Denote by Λ^k the powers of the operator Λ, for $k = 1, 2, \ldots$. Then, using inequality (3.21), after some algebraic calculations, we deduce that the operator $\Lambda^k : C([0,T];X) \to C([0,T];X)$ is a Lipschitz continuous operator with constant $L_k = \frac{C^kT^k}{k!}$, that is,

$$\|\Lambda^k\boldsymbol{\xi} - \Lambda^k\boldsymbol{\eta}\|_{C([0,T];X)} \leq L_k\|\boldsymbol{\xi} - \boldsymbol{\eta}\|_{C([0,T];X)} \quad \forall\, \boldsymbol{\xi},\, \boldsymbol{\eta} \in C([0,T];X). \tag{4.35}$$

Next, it is easy to see that

$$\lim_{k\to\infty} \frac{C^kT^k}{k!} = 0$$

and, therefore, inequality (4.35) guarantees that there exists $m \in \mathbb{N}$ such that Λ^m is a contraction on the space $C([0,T];X)$, i.e.,

$$L_m < 1. \tag{4.36}$$

We now use equality $\Lambda^m\boldsymbol{\theta} = \boldsymbol{\theta}$ (guaranteed by equality $\Lambda\boldsymbol{\theta} = \boldsymbol{\theta}$) to see that

$$\begin{aligned}\|\boldsymbol{\theta}_n - \boldsymbol{\theta}\|_{C([0,T];X)} &\leq \|\boldsymbol{\theta}_n - \Lambda\boldsymbol{\theta}_n\|_{C([0,T];X)} + \|\Lambda\boldsymbol{\theta}_n - \Lambda^2\boldsymbol{\theta}_n\|_{C([0,T];X)}\\ &+ \ldots + \|\Lambda^{m-1}\boldsymbol{\theta}_n - \Lambda^m\boldsymbol{\theta}_n\|_{C([0,T];X)} + \|\Lambda^m\boldsymbol{\theta}_n - \Lambda^m\boldsymbol{\theta}\|_{C([0,T];X)}.\end{aligned}$$

Next, using (4.35) we find that

$$\|\boldsymbol{\theta}_n - \boldsymbol{\theta}\|_{C([0,T];X)} \leq \big(1 + L_1 + L_2 + \ldots + L_{m-1}\big)\|\boldsymbol{\theta}_n - \Lambda\boldsymbol{\theta}_n\|_{C([0,T];X)} + L_m\|\boldsymbol{\theta}_n - \boldsymbol{\theta}\|_{C([0,T];X)},$$

which implies that

$$(1-L_m)\|\boldsymbol{\theta}_n-\boldsymbol{\theta}\|_{C([0,T];X)}\le \big(1+L_1+L_2+\ldots+L_{m-1}\big)\|\boldsymbol{\theta}_n-\Lambda\boldsymbol{\theta}_n\|_{C([0,T];X)}.$$

Then, inequality (4.36) yields

$$\|\boldsymbol{\theta}_n-\boldsymbol{\theta}\|_{C([0,T];X)}\le \frac{1}{1-L_m}\big(1+L_1+L_2+\ldots+L_{m-1}\big)\|\boldsymbol{\theta}_n-\Lambda\boldsymbol{\theta}_n\|_{C([0,T];X)}.$$

Finally, we use the convergence (4.34) to see that $\boldsymbol{\theta}_n\to\boldsymbol{\theta}$ in $C([0,T];X)$ and, therefore, the statement (S_1) holds. □

Remark 4.1. *Theorem 4.1 shows that a sequence $\{\boldsymbol{\theta}_n\}=\{(\boldsymbol{u}_n,\boldsymbol{\sigma}_n)\}\subset C([0,T];X)$ converges uniformly to the solution $\boldsymbol{\theta}=(\boldsymbol{u},\boldsymbol{\sigma})$ of Problem $\mathcal{P}_V$ if and only if it satisfies condition (S_2). It turns out from here that this condition represents a* criterion of convergence *to the solution of Problem $\mathcal{P}_V$.*

We now provide an application of Theorem 3.1. To this end, we keep assumption (2.14)–(2.19) and, for each $n\in\mathbb{N}$ we consider a function $\boldsymbol{f}_n\in C([0,T];V^*)$ as well a constant g_n such that

$$\boldsymbol{f}_n\in C([0,T];V^*), \tag{4.37}$$

$$g_n>0. \tag{4.38}$$

We denote by K_n the set defined by

$$K_n=\{\,\boldsymbol{v}\in V\,:\,v_\nu\le g_n\quad\text{a.e. on }\Gamma_3\,\} \tag{4.39}$$

and consider the following variational problem.

Problem $\mathcal{P}_V^n$. *Find a displacement field $\boldsymbol{u}_n:[0,T]\to K_n$ and a stress field $\boldsymbol{\sigma}_n:[0,T]\to Q$ such that*

$$(\boldsymbol{\sigma}_n(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q+\langle P\boldsymbol{u}_n(t),\boldsymbol{v}-\boldsymbol{u}_n(t)\rangle\ge\langle\boldsymbol{f}_n(t),\boldsymbol{v}-\boldsymbol{u}_n(t)\rangle\quad\forall\,\boldsymbol{v}\in K_n, \tag{4.40}$$

$$\boldsymbol{\sigma}_n(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}_n(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_n(s)))\,ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{4.41}$$

for all $t\in[0,T]$.

It follows from Theorem 3.1 that Problem $\mathcal{P}_V^n$ has a unique solution with regularity $\boldsymbol{u}_n\in C([0,T];K_n)$, $\boldsymbol{\sigma}_n\in C([0,T];Q)$. Assume now that the following convergences hold:

$$\boldsymbol{f}_n\to\boldsymbol{f}\quad\text{in}\quad C([0,T];V^*), \tag{4.42}$$

$$g_n\to g. \tag{4.43}$$

The link between the solutions of Problems $\mathcal{P}_V$ and $\mathcal{P}_V^n$ is given by the following result.

Theorem 4.2. *Assume (2.14)–(2.19), (4.37), (4.38), (4.42) and (4.43). Then,*

$$\boldsymbol{u}_n\to\boldsymbol{u}\quad\text{in}\quad C([0,T];V),\qquad\boldsymbol{\sigma}_n\to\boldsymbol{\sigma}\quad\text{in}\quad C([0,T];Q). \tag{4.44}$$

The proof of Theorem 4.2 requires some preliminary results. We assume in what follows that (2.14)–(2.19), (4.37), (4.38), hold and denote by $\mathcal{F}_n:V^*\to K_n$ the map defined by

$$\boldsymbol{z}=\mathcal{F}_n\boldsymbol{\omega}\iff\boldsymbol{z}\in K_n,\quad\langle A\boldsymbol{z},\boldsymbol{v}-\boldsymbol{z}\rangle\ge\langle\boldsymbol{\omega},\boldsymbol{v}-\boldsymbol{z}\rangle\quad\forall\,\boldsymbol{v}\in K_n, \tag{4.45}$$

for any $\boldsymbol{\omega}\in V^*$. We have the following result.

Lemma 4.2. *Let $\{\boldsymbol{\omega}_n\} \subset V^*$ be a bounded sequence, $\{g_n\} \subset \mathbb{R}$ a sequence such that $g_n \geq m_0$ for all $n \in \mathbb{N}$ with some $m_0 > 0$, $g > 0$, and let $\boldsymbol{\omega} \in V^*$. There exists a constant C which does not depend on $n \in \mathbb{N}$ and such that*

$$\|\mathcal{F}_n\boldsymbol{\omega}_n - \mathcal{F}\boldsymbol{\omega}\|_V \leq C(\|\boldsymbol{\omega}_n - \boldsymbol{\omega}\|_{V^*} + \sqrt{|g_n - g|}). \tag{4.46}$$

Proof. Let $\boldsymbol{\omega}_n$, $\boldsymbol{\omega} \in V^*$, $n \in \mathbb{N}$ and denote $\mathcal{F}_n\boldsymbol{\omega}_n = \boldsymbol{z}_n$, $\mathcal{F}\boldsymbol{\omega} = \boldsymbol{z}$. We use (3.6) and (4.45) to see that

$$\langle A\boldsymbol{z}_n, \boldsymbol{v} - \boldsymbol{z}_n\rangle \geq \langle \boldsymbol{\omega}_n, \boldsymbol{v} - \boldsymbol{z}_n\rangle \quad \forall\, \boldsymbol{v} \in K_n, \tag{4.47}$$

$$\langle A\boldsymbol{z}, \boldsymbol{v} - \boldsymbol{z}\rangle \geq \langle \boldsymbol{\omega}, \boldsymbol{v} - \boldsymbol{z}\rangle \quad \forall\, \boldsymbol{v} \in K. \tag{4.48}$$

Then, taking $\boldsymbol{v} = \mathbf{0}_V$ in (4.47) and using inequality (3.5) (a) together with equality $A\mathbf{0}_V = \mathbf{0}_{V^*}$, we find that

$$m_{\mathcal{E}}\|\boldsymbol{z}_n\|_V^2 \leq \langle \boldsymbol{\omega}_n, \boldsymbol{z}_n\rangle \leq \|\boldsymbol{\omega}_n\|_{V^*}\|\boldsymbol{z}_n\|_V,$$

which shows that the sequence $\{\boldsymbol{z}_n\} \subset V^*$ is bounded in V. Denote

$$\alpha_n = \frac{g_n}{g}, \quad \beta_n = \frac{g}{g_n}, \tag{4.49}$$

and note that $\alpha_n\boldsymbol{z} \in K_n$, $\beta_n\boldsymbol{z}_n \in K$. We take $\boldsymbol{v} = \alpha_n\boldsymbol{z}$ in (4.47) and $\boldsymbol{v} = \beta_n\boldsymbol{z}_n$ in (4.48), then we add the resulting inequalities to see that

$$\langle A\boldsymbol{z}_n, \alpha_n\boldsymbol{z} - \boldsymbol{z}_n\rangle + \langle A\boldsymbol{z}, \beta_n\boldsymbol{z}_n - \boldsymbol{z}\rangle \geq \langle \boldsymbol{\omega}_n, \alpha_n\boldsymbol{z} - \boldsymbol{z}_n\rangle + \langle \boldsymbol{\omega}, \beta_n\boldsymbol{z}_n - \boldsymbol{z}\rangle.$$

We now write $\alpha_n\boldsymbol{z} = \boldsymbol{z} + (\alpha_n - 1)\boldsymbol{z}$, $\beta_n\boldsymbol{z}_n = \boldsymbol{z}_n + (\beta_n - 1)\boldsymbol{z}_n$ to find that

$$\begin{aligned}\langle A\boldsymbol{z}_n - A\boldsymbol{z}, \boldsymbol{z}_n - \boldsymbol{z}\rangle &\leq (\alpha_n - 1)\Big[\langle A\boldsymbol{z}_n, \boldsymbol{z}\rangle - \langle \boldsymbol{\omega}_n, \boldsymbol{z}\rangle\Big] \\ &+(\beta_n - 1)\Big[\langle A\boldsymbol{z}, \boldsymbol{z}_n\rangle - \langle \boldsymbol{\omega}, \boldsymbol{z}_n\rangle\Big] + \langle \boldsymbol{\omega}_n - \boldsymbol{\omega}, \boldsymbol{z}_n - \boldsymbol{z}\rangle.\end{aligned}$$

Then, using the boundedness of the sequences $\{\boldsymbol{z}_n\} \subset V$, $\{\boldsymbol{\omega}_n\} \subset V^*$ and the properties of A, we find that there exists a positive constant C which does not depend on n such that

$$\|\boldsymbol{z}_n - \boldsymbol{z}\|_V^2 \leq C\|\boldsymbol{\omega}_n - \boldsymbol{\omega}\|_{V^*}\|\boldsymbol{z}_n - \boldsymbol{z}\|_V + C(|\alpha_n - 1| + |\beta_n - 1|).$$

We now use inequality (4.7) to find that

$$\|\boldsymbol{z}_n - \boldsymbol{z}\|_V \leq C\|\boldsymbol{\omega}_n - \boldsymbol{\omega}\|_{V^*} + \sqrt{C(|\alpha_n - 1| + |\beta_n - 1|)}. \tag{4.50}$$

Moreover, equalities (4.49) combined with inequality $g_n \geq m_0$ show that

$$|\alpha_n - 1| + |\beta_n - 1| \leq C|g_n - g|. \tag{4.51}$$

Finally, we combine inequalities (4.50) and (4.51) to see that (4.46) holds. □

We now provide the proof of Theorem 4.2.

Proof. Let $n \in \mathbb{N}$ and $t \in [0, T]$. It follows from the proof of Theorem 3.1 that the solution of Problem $\mathcal{P}_V^n$ satisfies the equalities

$$\boldsymbol{u}_n(t) = \mathcal{F}_n(\boldsymbol{f}_n(t) - \boldsymbol{\varepsilon}^*\Lambda_0\boldsymbol{\theta}_n(t)), \quad \boldsymbol{\sigma}_n(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) + \Lambda_0\boldsymbol{\theta}_n(t) \tag{4.52}$$

where, recall, the operator Λ_0 is given by (3.9). Then, using (3.10) and (4.46) we deduce that

$$\|\boldsymbol{u}_n(t) - \Lambda_1\boldsymbol{\theta}_n(t)\|_V \le C(\|\boldsymbol{f}_n(t) - \boldsymbol{f}(t)\|_{V^*} + \sqrt{|g_n - g|}). \tag{4.53}$$

On the other hand, using (4.52), (3.11) and (3.3) we see that

$$\|\boldsymbol{\sigma}_n(t) - \Lambda_2\boldsymbol{\theta}_n(t)\|_Q = \|\mathcal{E}\varepsilon(\boldsymbol{u}_n(t)) - \mathcal{E}\varepsilon(\Lambda_1\boldsymbol{\theta}_n(t))\|_Q \le M_{\mathcal{E}}\|\boldsymbol{u}_n(t) - \Lambda_1\boldsymbol{\theta}_n(t)\|_V$$

and, therefore, (4.53) implies that

$$\|\boldsymbol{\sigma}_n(t) - \Lambda_2\boldsymbol{\theta}_n(t)\|_Q \le C(\|\boldsymbol{f}_n(t) - \boldsymbol{f}(t)\|_{V^*} + \sqrt{|g_n - g|}). \tag{4.54}$$

Finally, we use the convergences (4.42) and (4.43) in (4.53) and (4.54) to deduce that the sequence $\{\boldsymbol{\theta}_n\} = \{(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)\}\}$ satisfies condition (S_3). Theorem 4.2 is now a consequence of Theorem 4.1. $\square$

We end this section with the remark that, besides the mathematical interest in the convergence result (4.44), it is interesting from the mechanical point of view, since it shows that the weak solution of the contact problem $\mathcal{P}$ depends continuously on the density of the applied forces and the thickness of the deformable layer.

5 Well-Posedness Results

The well-posedness concepts of a nonlinear problem depend on the problem considered, vary from author to author, and even from paper to paper. Nevertheless, most of them are based on two main ingredients: the existence and uniqueness of the solution to the corresponding problem and the convergence to it of a special class of sequences, the so-called approximating sequences. In this section, we introduce two well-posedness concepts in the study of Problem $\mathcal{P}_V$ and, to this end, we start with the following definition.

Definition 5.1. 1) *A sequence* $\{\boldsymbol{\theta}_n\} = \{(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)\} \subset C([0,T];X)$ *is said to be a* $\mathcal{T}_1$*-approximating sequence if there exists* $0 \le \varepsilon_n \to 0$ *such that*

$$\left\{\begin{array}{ll} \text{(a)} & \boldsymbol{u}_n(t) \in K, \\[2mm] \text{(b)} & (\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + \langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \\ & \qquad +\varepsilon_n\|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_V \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \quad \forall\, \boldsymbol{v} \in K, \\[2mm] \text{(c)} & \left\|\boldsymbol{\sigma}_n(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \displaystyle\int_0^t \mathcal{G}(\boldsymbol{\sigma}_n(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(s)))\,ds - \boldsymbol{\sigma}_0 + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\right\|_Q \le \varepsilon_n, \end{array}\right. \tag{5.1}$$

for all $n \in \mathbb{N}$ *and* $t \in [0,T]$.

2) *A sequence* $\{\boldsymbol{\theta}_n\} = \{(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)\} \subset C([0,T];X)$ *is said to be a* $\mathcal{T}_2$*-approximating sequence if there exists* $0 \le \varepsilon_n \to 0$ *such that* (4.1) *holds, that is, it satisfies condition* (S_2).

3) *Problem* $\mathcal{P}_V$ *is said to be* $\mathcal{T}_i$*-well-posed* $(i = 1, 2)$ *if it has a unique solution* $\boldsymbol{\theta} \in C([0,T];X)$ *and every* $\mathcal{T}_i$*-approximating sequence converges in* $C([0,T];X)$ *to* $\boldsymbol{\theta}$.

We now introduce the following notation, for $i = 1, 2$:

$$\mathcal{S} = \Big\{\, \{\boldsymbol{\theta}_n\} \subset C([0,T];X) \ :\ \boldsymbol{\theta}_n \to \boldsymbol{\theta} \ \text{ in } \ C([0,T];X) \,\Big\},$$

$$\mathcal{S}_i = \Big\{\, \{\boldsymbol{\theta}_n\} \subset C([0,T];X) \ : \{\boldsymbol{\theta}_n\} \text{ is a } \mathcal{T}_i\text{-approximating sequence} \,\Big\}.$$

Our main result in this section is the following.

Theorem 5.1. *Assume* (2.14)–(2.19). *Then, Problem $\mathcal{P}_V$ is $\mathcal{T}_1$- and $\mathcal{T}_2$-well-posed.*

Proof. First, we use Theorem 3.1 to see that Problem $\mathcal{P}_V$ has a unique solution. Next, we use Definition 5.1 2) and Theorem 4.1 to see that $\mathcal{S}_2 = \mathcal{S}$ which implies that $\mathcal{S}_2 \subset \mathcal{S}$. Therefore, Definition 5.1 3) guarantees that Problem $\mathcal{P}_V$ is $\mathcal{T}_2$-well-posed. Finally, Definition 5.1 1) shows that $\mathcal{S}_1 \subset \mathcal{S}_2$ and, since $\mathcal{S}_2 = \mathcal{S}$, we obtain that $\mathcal{S}_1 \subset \mathcal{S}$. We conclude from here that Problem $\mathcal{P}_V$ is $\mathcal{T}_1$-well-posed, too. □

A direct consequence of Theorem 5.1 is the following convergence result, which represents a particular case of Theorem 4.2.

Corollary 5.1. *Assume* (2.14)–(2.19), (4.37), (4.42) *and, moreover, assume that $g_n = g$, for each $n \in \mathbb{N}$. Then, the convergence* (4.44) *holds where, recall, $(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)$, $(\boldsymbol{u}, \boldsymbol{\sigma})$ represent the solutions of Problems $\mathcal{P}_V^n$ and $\mathcal{P}_V$, respectively.*

Proof. Let $n \in \mathbb{N}$ and $t \in [0, T]$. Since $g_n = g$, we deduce from (4.39) that $K_n = K$ and, therefore, (4.40) implies that

$$(\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q + \langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \geq \langle \boldsymbol{f}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \quad \forall\, \boldsymbol{v} \in K.$$

Hence,

$$\begin{aligned}(\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q &+ \langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \\ &+\langle \boldsymbol{f}(t) - \boldsymbol{f}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \quad \forall\, \boldsymbol{v} \in K\end{aligned}$$

and, moreover,

$$\begin{aligned}(\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))_Q &+ \langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \\ &+\|\boldsymbol{f}_n - \boldsymbol{f}\|_{C([0,T];V^*)}\|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_X \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \quad \forall\, \boldsymbol{v} \in K.\end{aligned} \tag{5.2}$$

Inequality (5.2) combined with equality (4.41) and inclusion $\boldsymbol{u}_n(t) \in K$ show that conditions (5.1) hold with $\varepsilon_n = \|\boldsymbol{f}_n - \boldsymbol{f}\|_{C([0;T]);V^*}$. Therefore, the convergence (4.42) implies that the sequence $\{\boldsymbol{\theta}_n\} = \{(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)\} \subset C([0, T]; X)$ a $\mathcal{T}_1$-approximating sequence. The convergence (4.44) is now a direct consequence of the inclusion $\mathcal{S}_1 \subset \mathcal{S}$, guaranteed by Theorem 5.1. □

We turn now to the inclusion $\mathcal{S}_1 \subset \mathcal{S}_2$ used in the proof of Theorem 5.1. The one-dimensional example below shows that this inclusion is strict, i.e., $\mathcal{S}_2 \neq \mathcal{S}_1$.

Example 5.1. *Consider Problem $\mathcal{P}$ in the particular case when $d = 1$, $\Omega = (0, 1)$, $\Gamma_1 = \{0\}$, $\Gamma_2 = \emptyset$, $\Gamma_3 = \{1\}$, $\mathcal{E} = 1$, $\mathcal{G}(\sigma, \varepsilon) = -\varepsilon$ for all $\varepsilon \in \mathbb{R}$, $f_0(x) = f$ for all $x \in [0, 1]$, $g > 0$, $p(r) = 0$ for all $r \in \mathbb{R}$, $\sigma_0(x) = f(1 - x)$, $u_0(x) = f\Big(x - \frac{x^2}{2}\Big)$ for each $x \in [0, 1]$, where $f > 0$ is a given constant. Note that in this particular case $\varepsilon(u) = u'$, where the prime represents the derivative with respect to the spatial variable x. Moreover,*

$$V = \{\, v \in H^1(0, 1) : \ v(0) = 0\}, \quad Q = L^2(0, 1),$$

$$K = \{\, v \in H^1(0, 1) : \ v(0) = 0, \quad v(1) \leq g\,\},$$

and recall that

$$(\sigma, \tau)_Q = \int_0^1 \sigma\tau\, dx \qquad \forall\, \sigma,\ \tau \in Q.$$

Then, using (2.28), (2.29), (2.25) and (2.24) it is easy to see that Problem $\mathcal{P}_V$ can be formulated as follows.

Problem $\widetilde{\mathcal{P}}_V$. *Find a couple of functions $u:[0,T]\to V$, $\sigma:[0,T]\to Q$ such that, for all $t\in[0,T]$, the following holds:*

$$\sigma(t)=u'(t)-\int_0^t u'(s)\,ds \qquad \forall\, t\in[0,T], \tag{5.3}$$

$$u(t)\in K, \quad \int_0^1 \sigma(t)(v'-u'(t))dx \geq \int_0^1 f(v-u(t))dx \quad \forall\, v\in K. \tag{5.4}$$

Now, for each $n\in\mathbb{N}$, we consider the problem (5.3), (5.4) with the data f_n and g_n. This problem can be formulated as follows.

Problem $\widetilde{\mathcal{P}}_V^n$. *Find a couple of functions $u_n:[0,T]\to V$, $\sigma_n:[0,T]\to Q$ such that, for all $t\in[0,T]$, the following holds:*

$$\sigma_n(t)=u_n'(t)-\int_0^t u_n'(s)\,ds \qquad \forall\, t\in[0,T], \tag{5.5}$$

$$u_n(t)\in K_n, \quad \int_0^1 \sigma_n(t)(v'-u_n'(t))dx \geq \int_0^1 f_n(v-u_n(t))dx \quad \forall\, v\in K_n. \tag{5.6}$$

Here, $K_n=\{\,v\in H^1(0,1):\ v(0)=0,\ \ v(1)\leq g_n\,\}$.

Assume now $T=ln\,\frac{2g}{f}$, $f_n=f+\frac{1}{n}$ and $g_n=g\Big(1+\frac{1}{fn}\Big)$. Then, it is easy to check that the solution of Problem $\widetilde{\mathcal{P}}_V$ is given by

$$u(x,t)=f\Big(x-\frac{x^2}{2}\Big)e^t, \quad \sigma(x,t)=f(1-x) \quad \forall\, x\in[0,1],\ t\in[0,T]. \tag{5.7}$$

Moreover, the solution of Problem $\widetilde{\mathcal{P}}_V^n$ is given by

$$u_n(x,t)=\Big(f+\frac{1}{n}\Big)\Big(x-\frac{x^2}{2}\Big)e^t, \quad \sigma_n(x,t)=\Big(f+\frac{1}{n}\Big)(1-x) \quad \forall\, x\in[0,1],\ t\in[0,T]. \tag{5.8}$$

This implies that

$$u_n(1,T)=\frac{1}{2}\Big(f+\frac{1}{n}\Big)e^T=\frac{1}{2}\Big(f+\frac{1}{n}\Big)\frac{2g}{f}=g\Big(1+\frac{1}{fn}\Big)>g,$$

for each $n\in\mathbb{N}$.

We conclude from here that $u_n(T)\notin K$ for each $n\in\mathbb{N}$, which shows that the sequence $\{\theta_n\}=\{(u_n,\sigma_n)\}$ is not a $\mathcal{T}_1$-approximating sequence for Problem $\widetilde{\mathcal{P}}_V$. Nevertheless, it is easy to see that

$$u_n\to u \quad \text{in} \quad C([0,T];V) \quad \text{and} \quad \sigma_n\to\sigma \quad \text{in} \quad C([0,T];Q), \tag{5.9}$$

which validate the convergence result in Theorem 4.2 and, moreover, it shows that the sequence $\{\theta_n\}=\{(u_n,\sigma_n)\}$ is a $\mathcal{T}_2$-approximating sequence for Problem $\widetilde{\mathcal{P}}_V$. Note that the convergence (5.9) cannot be obtained invoking the $\mathcal{T}_1$-well-posed of Problem $\widetilde{\mathcal{P}}_V$. We conclude from here that the concept of $\mathcal{T}_2$-well-posedness is stronger than concept of $\mathcal{T}_1$-well-posedness.

We end this section with the following comments on the equality $\mathcal{S}_2=\mathcal{S}$ above.

Remark 5.1. *We claim that among all the concepts which make Problem $\mathcal{P}_V$ well-posed, the $\mathcal{T}_2$-well-posedness concept in Definition 5.1 is optimal, in the sense that it uses the largest set of approximating sequences. Indeed, consider a different well-posedness concept, say the $\mathcal{T}$-well-posedness concept, defined by a set of $\mathcal{T}$-approximating sequences, denoted by $\mathcal{S}_{\mathcal{T}}$. Then, if Problem $\mathcal{P}_V$ is $\mathcal{T}$-well-posed we have $\mathcal{S}_{\mathcal{T}} \subset \mathcal{S}$, by definition. Now, since $\mathcal{S}_2 = \mathcal{S}$, we deduce that $\mathcal{S}_{\mathcal{T}} \subset \mathcal{S}_2$, which justifies our claim. In particular, the $\mathcal{T}_2$-well-posedness concept in Definition 5.1 is better than the $\mathcal{T}_1$-well-posedness concept in Definition 5.1.*

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