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Research Article

Infinitely Many Specific Multicomposite Deterministic and Fuzzy Trigonometric and Hyperbolic Neural Network Approximations

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Abstract: In this article we study in detail the multicomposite deterministic and fuzzy approximation to the unit by infinitely many multicomposite deterministic and fuzzy neural network operators activated by specific multicomposite sigmoid functions. The derived main results generalize earlier related trigonometric and hyperbolic complex valued ones into the deterministic and fuzzy levels. Here the high order multicomposite deterministic and fuzzy pointwise and uniform convergences with rates to the unit operator are given via multicomposite deterministic and fuzzy Jackson type inequalities, respectively. These involve the corresponding moduli of continuity of the mth order ($m \geq 0$) derivatives and H-fuzzy derivatives of the involved deterministic and fuzzy number valued functions, respectively. The treated operators are of averaged type. The derived results are univariate and multivariate.

Keywords: Deterministic and fuzzy overactivated trigonometric and hyperbolic neural network approximation, quasi-interpolation operator, precisely defined neural network operators.

AMS Math. Subject Class. (2020): 26A15, 26E50, 41A17, 41A25, 41A30, 47S40.

1 Introduction

Activation in Fuzzy Neural Networks

Fuzziness in neural networks refers to integrating fuzzy logic with artificial neural networks (ANNs) to handle uncertainty, imprecision, and vague data, enhancing both interpretability and learning capabilities. These hybrid models (neuro-fuzzy systems) use membership functions instead of crisp, binary inputs to represent human-like reasoning, allowing networks to process intermediate truth states rather than strict "true or false" values.

Key Features of Fuzzy Neural Networks

- **Fuzzy Components**: Fuzziness is implemented by defining neural weights, biases, or inputs as fuzzy sets (e.g., Low, Medium, High) rather than precise numbers.
- **Structure**: They typically include a fuzzification layer (input processing), a rule-based hidden layer (fuzzy inference), and a defuzzification layer (turning fuzzy results into a crisp output).

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- **Hybrid Strengths**: The system combines the learning, adaptation, and pattern recognition capabilities of neural networks with the human-like interpretability of fuzzy systems.
- **Reduced Complexity**: Some studies indicate that fuzzy networks can achieve high-quality results with smaller network architectures, aiding efficiency.

Applications and Benefits

Fuzzy NNs are ideal for systems where precise mathematical models are difficult to define.

- **Expert Systems & Control**: Used in modeling human decision-making and control systems.
- **Classification & Pattern recognition**: Capable of handling vague data, enhancing accuracy.
- **Interpretable Modeling**: Unlike "black-box" neural networks, these provide insights into how decisions are made.

While some consider traditional fuzzy logic less favored by modern industry compared to pure deep learning, fuzzy neural networks remain valuable for tasks requiring both high adaptability and explainable AI.

Multiactivation in Fuzzy Neural Networks

Fuzziness in multi-composite activation neural networks refers to the integration of **fuzzy logic** principles - which handle ambiguity and continuous, uncertain, or linguistic data - with traditional **deep neural networks** that use multiple, combined, or non-traditional activation functions to enhance learning, interpretability, and robustness. These networks are frequently referred to as **hybrid fuzzy-neural networks** or **neuro-fuzzy systems**.

Core Concepts of Fuzziness in Networks

- **Fuzzy Membership Functions**: Instead of relying on crisp (1 or 0) inputs, these networks use membership functions - such as Gaussian, S-shaped, or Z-shaped - to convert numerical data into a range, representing degrees of truth (e.g., "slow", "medium", "fast").
- **Fuzzy Rule Integration**: The hidden layers of these networks are often designed to implement fuzzy IF-THEN rules, allowing the systems to learn interpretable, human-readable logic while maintaining the adaptive power of neural networks.
- **Handling Uncertainty**: By utilizing fuzzy set membership, these models handle indistinct boundaries, noise, and overlapping classes better than conventional neural networks, making them effective for image processing or complex control systems with ambiguous data.

Multi-Composite Activation Features

- **Beyond ReLU:** While standard networks use ReLU, multi-composite activation networks often combine different functions (e.g., Sigmoid, Tanh, Leaky ReLU) in different layers or within the same neuron ensemble to capture diverse nonlinear relationships.
- **Overcoming "Dead Neurons":** The use of **LeakyReLU** as part of the composition helps alleviate the "dying ReLU" problem by allowing a small, non-zero gradient slope for negative inputs, keeping neurons active during training.

- **Fuzzy-guided Learning:** Fuzziness is used as a guidance mechanism, where a "fuzzy branch" provides supplementary information to multi-granular feature extraction modules, allowing the network to focus on essential features while ignoring noise.

Applications and Performance Benefits

- Image Segmentation & Classification: Fuzzy CNNs can better classify images with blurry boundaries by assigning pixels a degree of membership, reducing noise interference.
- **IoT & Fault Detection:** Fuzzy-CNN models have demonstrated superior accuracy in predicting mechanical faults, even in the presence of noise, by consolidating data through fuzzification layers.
- **Faster Convergence**: Fuzzy-neural networks with Ordered Fuzzy Numbers (OFN) or Type-2 fuzzy sets can lead to faster training and fewer required neurons to achieve the same accuracy as traditional networks, making them ideal for IoT devices.
- **Robust Control Systems:** These networks are used in intelligent controllers, such as those in vehicle braking systems, to manage nonlinear, uncertain dynamics.

Key Components of a Fuzzy System

1. **Fuzzification Layer:** Converts input data into fuzzy sets (using membership functions).
2. **Fuzzy Rule Computation Layer**: Processes rules (AND/OR operations) using fuzzy logic.
3. **Defuzzification Layer:** Translates the fuzzy output back into a precise numerical (crisp) value.

These networks are generally more interpretable than deep networks, as the internal rules can be translated into linguistic "if-then" descriptions.

In Section 2 we provide the necessary basic fuzzy background, in Section 3 we describe the essentials of deterministic multicomposite univariate and multivariate neural networks. Our main multicomposite univariate and multivariate deterministic convergence results come in Section 4. The corresponding main fuzzy results are contained in Section 5. This work is supported greatly by our monographs [1–5]. Interesting related research the reader can find in [6], [7–16].

2 Fuzzy Real Analysis Essentials

See also [2, Ch. 21, pp. 466–473]. We need the following background

Definition 2.1. (see [17]) *Let* $\mu : \mathbb{R} \to [0,1]$ *with the following properties*

(*i*) *is normal, i.e.,* $\exists\ x_0 \in \mathbb{R};\ \mu(x_0) = 1$.

(*ii*) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ *(*μ *is called a convex fuzzy subset).*

(*iii*) μ *is upper semicontinuous on* $\mathbb{R}$, *i.e.* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(*iv*) *The set* $\overline{\sup p(\mu)}$ *is compact in* $\mathbb{R}$, *(where* $\sup p(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$*).*

We call μ *a fuzzy real number. Denote the set of all* μ *with* $\mathbb{R}_{\mathcal{F}}$. *E.g.* $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, *for any* $x_0 \in \mathbb{R}$, *where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ (see [18]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\, r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [17]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_{\pm}^{(r)} = (\lambda \odot u)_{\pm}^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\}, \tag{2.1}$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \quad u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [17, 19].

Let $f, g : W \subseteq \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f,g) = \sup_{x \in W} D(f(x), g(x)).$$

Denote $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$.

Remark 2.1. *We determine and use*

$$D^*(f, \widetilde{o}) = \sup_{x \in W} D(f(x), \widetilde{o}) = \sup_{x \in W} \sup_{r \in [0,1]} \max\left\{\left|f_-^{(r)}(x)\right|, \left|f_+^{(r)}(x)\right|\right\}.$$

By the principle of iterated suprema we find that

$$D^*(f, \widetilde{o}) = \sup_{r \in [0,1]} \max\left\{\left\|f_-^{(r)}\right\|_\infty, \left\|f_+^{(r)}\right\|_\infty\right\}, \tag{2.2}$$

under the assumption $D^*(f, \widetilde{o}) < \infty$, that is f is a fuzzy bounded function. Above $\|\cdot\|_\infty$ is the supremum norm of the function over $W \subseteq \mathbb{R}^m$.

Here Σ^* stands for fuzzy summation and $\widetilde{0}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \forall\, u \in \mathbb{R}_{\mathcal{F}}.$$

Definition 2.2. *Let $f \in C(W)$, $W \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$, which is bounded or uniformly continuous, we define* $(h > 0)$

$$\omega_1(f,h) := \sup_{x,y \in W,\ \|x-y\|_\infty \leq h} |f(x) - f(y)|,$$

where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

Definition 2.3. *Let $f : W \to \mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^m$, we define the fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathcal{F})}(f,h) = \sup_{x,y \in W,\ \|x-y\|_\infty \leq h} D(f(x), f(y)), \quad h > 0.$$

where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

For $f : W \to \mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^m$, we use

$$[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right],$$

where $f_\pm^{(r)} : W \to \mathbb{R}$, $\forall\ r \in [0,1]$.

We need

Proposition 2.1. ([1, p. 246]) *Let $f : W \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f,\delta)$, $\omega_1\left(f_-^{(r)}, \delta\right)$, $\omega_1\left(f_+^{(r)}, \delta\right)$ are finite for any $\delta > 0$, $r \in [0,1]$. Then*

$$\omega_1^{(\mathcal{F})}(f,\delta) = \sup_{r \in [0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \delta\right), \omega_1\left(f_+^{(r)}, \delta\right)\right\}. \tag{2.3}$$

We denote by $C_{\mathcal{F}}^U(W)$ the space of fuzzy uniformly continuous functions from $W \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(W)$ is the space of fuzzy continuous functions on $W \subseteq \mathbb{R}^m$, and $C_B(W, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 2.2. ([1, Ch. 10]) *Let $f \in C_{\mathcal{F}}^U(W)$, where $W \subseteq \mathbb{R}^m$ is convex. Then $\omega_1^{(\mathcal{F})}(f,\delta) < \infty$, for any $\delta > 0$.*

Proposition 2.3. ([1, Ch. 10]) *It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f,\delta) = \omega_1^{(\mathcal{F})}(f,0) = 0,$$

iff $f \in C_{\mathcal{F}}^U(W)$, $W \subseteq \mathbb{R}^m$, where W is convex and compact.

Proposition 2.4. ([1, Ch. 10]) *Let $f \in C_{\mathcal{F}}(W)$, $W \subseteq \mathbb{R}^m$ open or compact. Then $f_\pm^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over W, respectively in $\pm$.*

Note 2.1. *It is clear by Propositions 2.1 and 2.3, that if $f \in C_{\mathcal{F}}^U(W)$, then $f_\pm^{(r)} \in C_U(W)$ (uniformly continuous on W). Also if $f \in C_B(W, \mathbb{R}_{\mathcal{F}})$, it implies by (2.2) that $f_\pm^{(r)} \in C_B(W)$ (continuous and bounded functions on W).*

We need

Definition 2.4. *Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.*

Definition 2.5. ([17, 20]) *Let $T := [x_0, x_0+\beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)*

$$\lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h\to 0+} \frac{f(x) - f(x-h)}{h}$$

exist and are equal to $f'(x)$. We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exist in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Definition 2.6. *We denote by $C_{\mathcal{F}}^{N^*}(W)$, $N^* \in \mathbb{N}$, the space of all N^*-times fuzzy continuously differentiable functions from W into $\mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^m$ open or compact which is convex.*

Here fuzzy partial derivatives are defined via Definition 2.5 in the obvious way as in the ordinary real case. We mention

Theorem 2.1. ([21]) *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in [a,b]$, $0 \le r \le 1$. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ are differentiable and

$$\left[f'(t)\right]^r = \left[\left(f(t)_-^{(r)} \right)', \left(f(t)_+^{(r)} \right)' \right].$$

I.e.

$$\left(f'\right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)', \quad \forall\, r \in [0,1]. \tag{2.4}$$

Remark 2.2. (see also [1, Ch. 11]) *Let $f \in C^{N^*}([a,b], \mathbb{R}_{\mathcal{F}})$, $N^* \ge 1$. Then by Theorem 2.1 we obtain $f_{\pm}^{(r)} \in C^{N^*}([a,b])$ and*

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)_-^{(r)} \right)^{(i)}, \left(f(t)_+^{(r)} \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N^*$, and in particular we have

$$\left(f^{(i)} \right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)^{(i)}, \tag{2.5}$$

for any $r \in [0,1]$.

Let $f \in C_{\mathcal{F}}^{N^*}(W)$, $W \subseteq \mathbb{R}^m$, open or compact, which is convex, denote $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\widetilde{\alpha_1}, ..., \widetilde{\alpha_m})$, $\widetilde{\alpha_i} \in \mathbb{Z}^+$, $i = 1, ..., m$ and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^{m} \widetilde{\alpha_i} \le N^*, \quad N^* > 1.$$

Then by Theorem 2.1 we get that

$$\left(f_{\pm}^{(r)} \right)_{\widetilde{\alpha}} = \left(f_{\widetilde{\alpha}} \right)_{\pm}^{(r)}, \quad \forall\, r \in [0,1],$$

and any $\widetilde{\alpha} : |\widetilde{\alpha}| \le N^*$. Here $f_{\pm}^{(r)} \in C^{N^*}(W)$.

Notation 2.1. *We denote*

$$\left(\sum_{i=1}^{2} D\left(\frac{\partial}{\partial x_i},\widetilde{0}\right)\right)^2 f\left(\overrightarrow{x}\right) := D\left(\frac{\partial^2 f\left(x_1,x_2\right)}{\partial x_1^2},\widetilde{0}\right)+D\left(\frac{\partial^2 f\left(x_1,x_2\right)}{\partial x_2^2},\widetilde{0}\right)+2D\left(\frac{\partial^2 f\left(x_1,x_2\right)}{\partial x_1\partial x_2},\widetilde{0}\right).$$

In general we denote $(j=1,...,N^*)$

$$\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i},\widetilde{0}\right)\right)^j f\left(\overrightarrow{x}\right) := \sum_{(j_1,...,j_m)\in\mathbb{Z}_+^m:\sum_{i=1}^m j_i=j} \frac{j!}{j_1!j_2!...j_m!} D\left(\frac{\partial^j f\left(x_1,...,x_m\right)}{\partial x_1^{j_1}\partial x_2^{j_2}...\partial x_m^{j_m}},\widetilde{0}\right).$$

3 Foundations of Multicomposite Univariate and Multivariate Neural Networks

Definition 3.1. ([4, p. 530]) *The general sigmoid activation functions we deal with here are described as follows* $h:\mathbb{R}\to[-1,1]$, *strictly increasing* $h(0)=0$, $h(-x)=-h(x)$, $x\in\mathbb{R}$, $h(+\infty)=1$, $h(-\infty)=-1$. *Also* h *is strictly convex over* $(-\infty,0]$ *and strictly concave over* $[0,+\infty)$, *with* $h^{(2)}\in C(\mathbb{R})$.

More precisely we will deal with the following sigmoid activation functions which possess all of the above properties: the algebraic activation function,

$$g_1(x)=\frac{x}{\sqrt[2m]{1+x^{2m}}},\quad m\in\mathbb{N},\ x\in\mathbb{R},$$

(see [3, pp. 2–3]); the normalized and parametrized arctangent

$$g_2(x)=\frac{2}{\pi}\arctan\left(\frac{x}{2}\lambda x\right)=\frac{2}{\pi}\int_0^{\frac{\pi\lambda x}{2}}\frac{dz}{1+z^2},\quad x\in\mathbb{R},\ \lambda>0,$$

(see [4, pp. 156–157]); the normalized and parametrized Gudermannian function

$$g_3(x)=\frac{2}{\pi}gd(\lambda x)=\frac{4}{\pi}\arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right)$$
$$=\frac{2}{\pi}\int_0^{\lambda x}\frac{dt}{\cosh t}=\frac{4}{\pi}\int_0^{\lambda x}\frac{dt}{e^t+e^{-t}},\quad x\in\mathbb{R},\ \lambda>0,$$

where

$$gd(x):=\int_0^x\frac{dt}{\cosh t}=2\arctan\left(\tanh\left(\frac{x}{2}\right)\right),\quad\forall\ x\in\mathbb{R},$$

is the Gudermannian function (see [4, pp. 198–199]); the parametrized error function

$$g_4(x)=erf\,\lambda x=\frac{2}{\sqrt{\pi}}\int_0^{\lambda x}e^{-t^2}dt,\quad x\in\mathbb{R},\ \lambda>0,$$

(see [4, pp. 226–227]); the parametrized hyperbolic tangent function

$$g_5(x)=\tanh\lambda x=\frac{e^{\lambda x}-e^{-\lambda x}}{e^{\lambda x}+e^{-\lambda x}}=\frac{e^{2\lambda x}-1}{e^{2\lambda x}+1}=\frac{1-e^{-2\lambda x}}{1+e^{-2\lambda x}},\quad x>0,\ \lambda>0,$$

(see [4, p. 72]); for small $0 < \lambda < 1$, $\tanh \lambda x$ is expected to behave better than *ReLu* and *Leaky ReLu* activation functions; the parametrized hyperbolic tangent like activation function,

$$g_6(x) = \frac{A^{\lambda x} - A^{-\lambda x}}{A^{\lambda x} + A^{-\lambda x}}, \quad A > 1,\ \lambda > 0,\ x \in \mathbb{R}$$

(see [4, pp. 262–263]); finally, the β-parametrized half hyperbolic tangent function

$$g_7(x) = \frac{1 - e^{-\beta x}}{1 + e^{-\beta x}}, \quad \beta > 0,\ \forall\ x \in \mathbb{R},$$

(see [4, pp. 509–510]). So, we denote them by $g_i(x)$, $x \in \mathbb{R}$, $i = 1, 2, ..., 7$. Here we deal with the specific sequences of activation functions $\left(h_i^{(q)}\right)$, $q \in \mathbb{N}$, $i \in \mathbb{N}$, such that all $h_i^{(q)} \in G := \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$. That is $h_i^{(q)}$ for different $i \in \mathbb{N}$ could be identical, $q \in \mathbb{N}$.

Therefore in the sequences of activation functions $h_1^{(q)}, h_2^{(q)}, ...$, we allow random repetitions. Thus any two random $h_i^{(q)}$ activation functions are not necessarily different, but all belong to G. Call $H^{(q)} = \left\{h_1^{(q)}, h_2^{(q)}, ...\right\}$, $q \in \mathbb{N}$, and by $\Omega := \left\{H^{(q)},\ \text{all } q \in \mathbb{N}\right\}$. For simplicity next we can consider such a sequence $\{h_1, h_2, ...\}$.

Notice here $0 < h_i(1) \leq 1$, $i = 1, 2, ...$ Any composition $h_1 \circ h_2 \circ h_3 \circ ... \circ h_\lambda$ is meant to be $h_1|_{[-1,1]} \circ h_2|_{[-1,1]} \circ h_3|_{[-1,1]} \circ ... \circ h_{\lambda-1}|_{[-1,1]} \circ h_\lambda$, $\lambda \in \mathbb{N}$, and for convenience, we denote it by $G_\lambda := h_1 \circ h_2 \circ h_3 \circ ... \circ h_\lambda$. We have for any $\lambda \in \mathbb{N} : 0 < h_\lambda(1) \leq 1$, hence $0 < h_{\lambda-1}(h_\lambda(1)) \leq h_{\lambda-1}(1) \leq 1$, and $0 < h_{\lambda-2}(h_{\lambda-1}(h_\lambda(1))) \leq h_{\lambda-2}(h_{\lambda-1}(1)) \leq h_{\lambda-2}(1) \leq 1$.

Inductively we derive that $0 < G_\lambda(1) \leq 1$, $\forall\ \lambda \in \mathbb{N}$. Clearly, it is $G_\lambda(0) = 0$ and G_λ is strictly increasing over $\mathbb{R}$. Furthermore it holds

$$\begin{aligned} G_\lambda(-x) &= h_1\Big(h_2\big(h_3\big(\cdots h_{\lambda-1}(h_\lambda(-x))\cdots\big)\big)\Big) \\ &= h_1\Big(h_2\big(h_3\big(\cdots h_{\lambda-1}(-h_\lambda(x))\cdots\big)\big)\Big) \\ &= \cdots \\ &= -h_1\Big(h_2\big(h_3\big(\cdots h_{\lambda-1}(h_\lambda(x))\cdots\big)\big)\Big) \\ &= -G_\lambda(x), \qquad x \in \mathbb{R}. \end{aligned}$$

Clearly it holds $G_\lambda^{(2)} \in C(\mathbb{R})$. We notice that

$$G_\lambda(+\infty) = h_1(h_2(h_3(...(h_{\lambda-1}(h_\lambda(+\infty)))))) = h_1(h_2(h_3(...(h_{\lambda-1}(1))))) = G_{\lambda-1}(1),$$

and

$$\begin{aligned} G_\lambda(-\infty) &= h_1\Big(h_2\big(h_3\big(\cdots h_{\lambda-1}(h_\lambda(-\infty))\cdots\big)\big)\Big) \\ &= h_1\Big(h_2\big(h_3\big(\cdots h_{\lambda-1}(-1)\cdots\big)\big)\Big) \\ &= -h_1\Big(h_2\big(h_3\big(\cdots h_{\lambda-1}(1)\cdots\big)\big)\Big) \\ &= -G_{\lambda-1}(1). \end{aligned}$$

Consequently, it holds

$$-G_{\lambda-1}(1) \leq G_\lambda(x) \leq G_{\lambda-1}(1), \quad \forall\ x \in \mathbb{R}.$$

Thus, $y = \pm G_{\lambda-1}(1)$ are asymptotes of $G_\lambda(x)$, any $\lambda \in \mathbb{N}$.

Next we act over $(-\infty,0]$: let $\lambda,\mu\geq 0:\lambda+\mu=1$. Then by convexity of h_2 there we have

$$h_2\left(\lambda x+\mu y\right)\leq \lambda h_2\left(x\right)+\mu h_2\left(y\right),\ \ \forall\ x,y\in(-\infty,0];$$

and

$$h_1\left(h_2\left(\lambda x+\mu y\right)\right)\leq h_1\left(\lambda h_2\left(x\right)+\mu h_2\left(y\right)\right)\leq \lambda\left(h_1\circ h_2\right)\left(x\right)+\mu\left(h_1\circ h_2\right)\left(y\right),\ \ \forall x,\ y\in(-\infty,0].$$

So that $h_1\circ h_2$ is convex over $(-\infty,0]$. Now we work on $[0,+\infty)$: let $\lambda,\mu\geq 0:\lambda+\mu=1$. Then by concavity of h_2 there we have

$$h_2\left(\lambda x+\mu y\right)\geq \lambda h_2\left(x\right)+\mu h_2\left(y\right),\ \ \forall\ x,y\in[0,+\infty);$$

and

$$h_1\left(h_2\left(\lambda x+\mu y\right)\right)\geq h_1\left(\lambda h_2\left(x\right)+\mu h_2\left(y\right)\right)\geq \lambda\left(h_1\circ h_2\right)\left(x\right)+\mu\left(h_1\circ h_2\right)\left(y\right),\ \ \forall x,\ y\in[0,+\infty).$$

Thus, $h_1\circ h_2$ is concave over $[0,+\infty)$. Therefore $G_2=h_1\circ h_2$ is a general sigmoid activation function with asymptotes $y=\pm h_1\left(1\right)$, and fulfilling the rest of conditions of Definition 3.1.

Arguing as above $h_2\circ h_3:\mathbb{R}\to[-1,1]$, fulfills Definition 3.1, and $h_1\circ h_2\circ h_3$ does the same with asymptotes $h=\pm h_1\left(h_2\left(1\right)\right)$. Inductively, we prove that G_λ fulfills Definition 3.1 with asymptotes $y=\pm G_{\lambda-1}\left(1\right)$. We have established the following:

Theorem 3.1. *Let $q,\lambda\in\mathbb{N}$. Then $G_\lambda^{(q)}:=h_1^{(q)}\circ h_2^{(q)}\circ h_3^{(q)}\circ...\circ h_\lambda^{(q)}$ fulfill all the properties of Definition 3.1 with asymptotes $y=\pm G_{\lambda-1}^{(q)}\left(1\right)$. That is $G_\lambda^{(q)}$ are multi-composite general sigmoid activation functions from $\mathbb{R}\to[-1,1]$.*

Corollary 3.1. $\frac{G_\lambda^{(q)}}{G_{\lambda-1}^{(q)}(1)}$ *fulfill Definition 3.1 with asymptotes $y=\pm 1$.*

We call

$$\widetilde{G}_\lambda^{(q)}:=\frac{G_\lambda^{(q)}}{G_{\lambda-1}^{(q)}\left(1\right)}.$$

Remark 3.1. *Next we consider the functions*

$$T_\lambda^{(q)}\left(x\right):=\frac{1}{4}\left(\widetilde{G}_\lambda^{(q)}\left(x+1\right)-\widetilde{G}_\lambda^{(q)}\left(x-1\right)\right)>0,\ \ \forall\ x\in\mathbb{R},\ q,\lambda\in\mathbb{N}.$$

We observe that

$$\begin{aligned}
T_\lambda^{(q)}(-x)&=\frac{1}{4}\left(\widetilde{G}_\lambda^{(q)}(-x+1)-\widetilde{G}_\lambda^{(q)}(-x-1)\right)\\
&=\frac{1}{4}\left(\widetilde{G}_\lambda^{(q)}\big(-(x-1)\big)-\widetilde{G}_\lambda^{(q)}\big(-(x+1)\big)\right)\\
&=\frac{1}{4}\left(-\widetilde{G}_\lambda^{(q)}(x-1)+\widetilde{G}_\lambda^{(q)}(x+1)\right)\\
&=\frac{1}{4}\left(\widetilde{G}_\lambda^{(q)}(x+1)-\widetilde{G}_\lambda^{(q)}(x-1)\right)\\
&=T_\lambda^{(q)}(x).
\end{aligned}$$

That is $T_\lambda^{(q)}$ are even functions,

$$T_\lambda^{(q)}\left(-x\right)=T_\lambda^{(q)}\left(x\right),\ \ \forall\ x\in\mathbb{R},\ q,\lambda\in\mathbb{N}.$$

We see that

$$T_\lambda^{(q)}(0)=\frac{\widetilde{G}_\lambda^{(q)}(1)}{2}.$$

Let $x>1$, we have that

$$T_\lambda^{(q)\prime}(x)=\frac{1}{4}\left(\widetilde{G}_\lambda^{(q)\prime}(x+1)-\widetilde{G}_\lambda^{(q)\prime}(x-1)\right)<0,$$

by $\widetilde{G}_\lambda^{(q)\prime}$ being strictly decreasing over $[0,+\infty)$. Let now $0<x<1$, then $1-x>0$ and $0<1-x<1+x$. It holds $\widetilde{G}_\lambda^{(q)\prime}(x-1)=\widetilde{G}_\lambda^{(q)\prime}(1-x)>\widetilde{G}_\lambda^{(q)\prime}(x+1)$, so that again $T_\lambda^{(q)\prime}(x)<0$. Consequently $T_\lambda^{(q)}$ is strictly decreasing on $(0,+\infty)$. Clearly, $T_\lambda^{(q)}$ is strictly increasing on $(-\infty,0)$, and $T_\lambda^{(q)\prime}(0)=0$. Observe that

$$\lim_{x\to+\infty}T_\lambda^{(q)}(x)=\frac{1}{4}\left(\widetilde{G}_\lambda^{(q)}(+\infty)-\widetilde{G}_\lambda^{(q)}(+\infty)\right)=0,$$

and

$$\lim_{x\to-\infty}T_\lambda^{(q)}(x)=\frac{1}{4}\left(\widetilde{G}_\lambda^{(q)}(-\infty)-\widetilde{G}_\lambda^{(q)}(-\infty)\right)=0,\quad q,\lambda\in\mathbb{N}.$$

That is the x-axis is the horizontal asymptote on $T_\lambda^{(q)}$. As a result $T_\lambda^{(q)}$ is a bell shaped symmetric function with maximum

$$T_\lambda^{(q)}(0)=\frac{\widetilde{G}_\lambda^{(q)}(1)}{2}.$$

We need

Theorem 3.2. *It holds*

$$\sum_{i=-\infty}^{\infty}T_\lambda^{(q)}(x-i)=1,\ \ \forall\ x\in\mathbb{R};\quad q,\lambda\in\mathbb{N}. \tag{3.1}$$

Proof. As similar to [4, p. 286] is omitted. □

Theorem 3.3. *We have that*

$$\int_{-\infty}^{\infty}T_\lambda^{(q)}(x)\,dx=1,\quad q,\lambda\in\mathbb{N}. \tag{3.2}$$

Proof. As similar to [4, p. 287], it is omitted. □

So that $T_\lambda^{(q)}(x)$ can serve as a density function in general. We need

Theorem 3.4. *Let $0<\alpha<1$, and $n\in\mathbb{N}$ with $n^{1-\alpha}>2$. Then*

$$\sum_{\substack{k=-\infty\\ :\,|nx-k|\geq n^{1-\alpha}}}^{\infty}T_\lambda^{(q)}(nx-k)<\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\alpha}-2\right)\right), \tag{3.3}$$

and

$$\lim_{n\to+\infty}\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\alpha}-2\right)\right)=0.$$

Proof. As in [4, pp. 531–532], it is omitted. □

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number. We also need

Theorem 3.5. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_\lambda^{(q)}(nx-k)} < \frac{1}{T_\lambda^{(q)}(1)}, \quad \forall\, x \in [a,b]\,; \quad q, \lambda \in \mathbb{N}.$$

Proof. As similar to [4, p. 289], it is omitted. □

Remark 3.2. *We have that*

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_\lambda^{(q)}(nx-k) \neq 1; \quad q, \lambda \in \mathbb{N},$$

for at least some $x \in [a,b]$. See [4, p. 290], same reasoning.

Note 3.1. *For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by (3.1))*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_\lambda^{(q)}(nx-k) \leq 1.$$

We make

Remark 3.3. *We define*

$$\widehat{Z}_\lambda^{(q)}(x_1,...,x_N) := \widehat{Z}_\lambda^{(q)}(x) := \prod_{i=1}^{N} T_\lambda^{(q)}(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \; q, \lambda, N \in \mathbb{N}.$$

It has the properties:

(*i*)

$$\widehat{Z}_\lambda^{(q)}(x) > 0, \; \forall\, x \in \mathbb{R}^N,$$

(*ii*)

$$\begin{aligned}\sum_{k\in\mathbb{Z}^N} \widehat{Z}_\lambda^{(q)}(x-k) &:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \widehat{Z}_\lambda^{(q)}(x_1-k_1,\ldots,x_N-k_N) \\ &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \prod_{i=1}^{N} T_\lambda^{(q)}(x_i-k_i) \\ &= \prod_{i=1}^{N} \left(\sum_{k_i=-\infty}^{\infty} T_\lambda^{(q)}(x_i-k_i) \right) \\ &\overset{(3.1)}{=} 1.\end{aligned}$$

Hence

$$\sum_{k=-\infty}^{\infty} \widehat{Z}_\lambda^{(q)}(x-k) = 1, \quad q, \lambda \in \mathbb{N}.$$

That is

(*iii*)

$$\sum_{k=-\infty}^{\infty} \widehat{Z}_{\lambda}^{(q)}(nx-k)=1, \quad \forall x\in\mathbb{R}^N;\ q,\lambda,n\in\mathbb{N}.$$

And

(*iv*)

$$\int_{\mathbb{R}^N}\widehat{Z}_{\lambda}^{(q)}(x)\,dx=\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}T_{\lambda}^{(q)}(x_i)\right)dx_1...dx_N=\prod_{i=1}^{N}\left(\int_{-\infty}^{\infty}T_{\lambda}^{(q)}(x_i)\,dx_i\right)\overset{(3.2)}{=}1,$$

thus

$$\int_{\mathbb{R}^N}\widehat{Z}_{\lambda}^{(q)}(x)\,dx=1,$$

that is $\widehat{Z}_{\lambda}^{(q)}$ *is a multivariate density function,* $q,\lambda\in\mathbb{N}$. *Here denote* $x=(x_1,...,x_N)$, $\|x\|_{\infty}:=\max\{|x_1|,...,|x_N|\}$, $x\in\mathbb{R}^N$, *also set* $\infty:=(\infty,...,\infty)$, $-\infty:=(-\infty,...,-\infty)$ *upon the multivariate context,* $0<\beta<1$,

(*v*)

$$\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}\widehat{Z}_{\lambda}^{(q)}(nx-k)=\sum_{k_1=-\infty}^{\infty}...\sum_{\substack{k_N=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}\left(\prod_{i=1}^{N}T_{\lambda}^{(q)}(nx_i-k_i)\right)$$

$$=\prod_{i=1}^{N}\left(\sum_{\substack{k_i=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}T_{\lambda}^{(q)}(nx_i-k_i)\right)\quad (\text{for some } r_*\in\{1,...,N\})$$

$$\leq\left(\prod_{\substack{i=1\\ i\neq r_*}}^{N}\left(\sum_{k_i=-\infty}^{\infty}T_{\lambda}^{(q)}(nx_i-k_i)\right)\right)\left(\sum_{\substack{k_{r_*}=-\infty\\ \left|\frac{k_{r_*}}{n}-x_{r_*}\right|>\frac{1}{n^{\beta}}}}^{\infty}T_{\lambda}^{(q)}(nx_{r_*}-k_{r_*})\right)$$

$$=\sum_{\substack{k_{r_*}=-\infty\\ \left|\frac{k_{r_*}}{n}-x_{r_*}\right|>\frac{1}{n^{\beta}}}}^{\infty}T_{\lambda}^{(q)}(nx_r-k_r)=\sum_{\substack{k_{r_*}=-\infty\\ |nx_{r_*}-k_{r_*}|>n^{1-\beta}}}^{\infty}T_{\lambda}^{(q)}(nx_{r_*}-k_{r_*})$$

$$\overset{(3.3)}{<}1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right).$$

That is

$$\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}\widehat{Z}_{\lambda}^{(q)}(nx-k)<1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right),\ \text{as } n\to+\infty,$$

$0<\beta<1$, $q,\lambda,n\in\mathbb{N}: n^{1-\beta}>2$, $\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$.

By [4, p. 634, see (25.23) therein], we have

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widehat{Z}_{\lambda}^{(q)}(nx-k)} < \frac{1}{\left(T_{\lambda}^{(q)}(1)\right)^{N}},$$

$\forall\, x \in \prod_{i=1}^{N}[a_i, b_i]$, $n \in \mathbb{N}$. Above we denote $\lceil na \rceil := (\lceil na_1 \rceil, ..., \lceil na_N \rceil)$, $\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor)$, where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$. Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widehat{Z}_{\lambda}^{(q)}(nx-k) \neq 1,$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

Let $f \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$; $q, \lambda \in \mathbb{N}$. We define the multivariate multicomposite averaged positive linear neural network operators ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$):

$$\begin{aligned} &{}_{\lambda}^{(q)}A_n(f, x_1, ..., x_N) := {}_{\lambda}^{(q)}A_n(f, x) \\ &:= \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \widehat{Z}_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widehat{Z}_{\lambda}^{(q)}(nx-k)} \\ &= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} T_{\lambda}^{(q)}(nx_i - k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} T_{\lambda}^{(q)}(nx_i - k_i)\right)}. \end{aligned} \tag{3.4}$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

Let $f \in C^m\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., N$, and $|\alpha| := \sum_{i=1}^{N} \alpha_i = l$, where $l = 0, 1, ..., m$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l. We denote

$$\omega_{1,m}^{\max}(f_\alpha, h) := \max_{\alpha:|\alpha|=m} \omega_1(f_\alpha, h).$$

Call also

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{\alpha:|\alpha|=m} \{\|f_\alpha\|_\infty\},$$

where $\|\cdot\|_\infty$ is the supremum norm. In one dimension we define:

Definition 3.2. *Let $f \in C([a, b])$ and $q, \lambda, n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce the multicomposite linear neural network operator*

$${}_{\lambda}^{(q)}A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) T_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k)}, \quad x \in [a, b].$$

Notice that ${}_{\lambda}^{(q)}A_n(1) = 1$, for all $q, \lambda \in \mathbb{N}$ and it is a positive linear operator.

Clearly here ${}_{\lambda}^{(q)}A_n(f, x) \in C([a, b])$.

4 Univariate and Multivariate Multicomposite Deterministic Approximation Results

We present the following univariate results.

Theorem 4.1. *Let $f \in C^2([a,b],\mathbb{C})$, $0<\alpha<1$, $q,\lambda,n\in\mathbb{N}: n^{1-\alpha}>2$, $x\in[a,b]$. Then*

(1)

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right| \le \frac{1}{T_{\lambda}^{(q)}(1)}\left[\left|f'(x)\right|\left(\frac{1}{n^{\alpha}}+(b-a)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.$$
$$+\frac{|f''(x)|}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)$$
$$\left.+\left(\frac{\omega_1\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''+f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right].$$

(2) *if $f'(x)=f''(x)=0$, we obtain*

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right| \le \frac{1}{T_{\lambda}^{(q)}(1)}\left[\frac{\omega_1\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''+f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right],$$

notice here the high rate of convergence at $n^{-3\alpha}$,

(3) *furthermore we get*

$$\left\|{}_{\lambda}^{(q)}A_nf-f\right\|_{\infty} \le \frac{1}{T_{\lambda}^{(q)}(1)}\left[\left\|f'\right\|_{\infty}\left(\frac{1}{n^{\alpha}}+(b-a)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.$$
$$+\frac{\|f''\|_{\infty}}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)$$
$$\left.+\left(\frac{\omega_1\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''+f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right].$$

i.e. $\lim_{n\to+\infty}{}_{\lambda}^{(q)}A_n(f)=f$, pointwise and uniformly,

(4) *and finally, it holds*

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f'(x){}_{\lambda}^{(q)}A_n(\sin(\cdot-x),x)-2f''(x){}_{\lambda}^{(q)}A_n\left(\sin^2\left(\frac{\cdot-x}{2}\right),x\right)-f(x)\right|$$
$$\le \frac{1}{T_{\lambda}^{(q)}(1)}\left[\frac{\omega_1\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''+f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].$$

again here we achieve high speed of convergence at $n^{-3\alpha}$.

Proof. As similar to [5, pp. 187–188], it is omitted. □

We continue with the related hyperbolic high order neural network approximation.

Theorem 4.2. *Let $f \in C^2([a,b],\mathbb{C})$, $0<\alpha<1$, $q,\lambda,n\in\mathbb{N}: n^{1-\alpha}>2$, $x\in[a,b]$. Then*

(1)

$$\left|{}_{\lambda}^{(q)}A_n\left(f,x\right)-f\left(x\right)\right|\leq\frac{1}{T_{\lambda}^{(q)}\left(1\right)}\cosh\left(b-a\right)\left[\left|f'\left(x\right)\right|\left(\frac{1}{n^{\alpha}}+\left(b-a\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.$$
$$+\frac{\left|f''\left(x\right)\right|}{2}\left(\frac{1}{n^{2\alpha}}+\left(b-a\right)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)$$
$$\left.+\left(\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''-f\right\|_{\infty}\left(b-a\right)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right].$$

(2) *if* $f'\left(x\right)=f''\left(x\right)=0$, *we obtain*

$$\left|{}_{\lambda}^{(q)}A_n\left(f,x\right)-f\left(x\right)\right|\leq\frac{1}{T_{\lambda}^{(q)}\left(1\right)}\cosh\left(b-a\right)$$
$$\times\left[\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''-f\right\|_{\infty}\left(b-a\right)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].$$

notice here the high rate of convergence at $n^{-3\alpha}$,

(3) *furthermore, we get*

$$\left\|{}_{\lambda}^{(q)}A_nf-f\right\|_{\infty}\leq\frac{1}{T_{\lambda}^{(q)}\left(1\right)}\cosh\left(b-a\right)\left[\left\|f'\right\|_{\infty}\left(\frac{1}{n^{\alpha}}+\left(b-a\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.$$
$$+\frac{\left\|f''\right\|_{\infty}}{2}\left(\frac{1}{n^{2\alpha}}+\left(b-a\right)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)$$
$$\left.+\left(\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''-f\right\|_{\infty}\left(b-a\right)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right].$$

it follows that $\lim_{n\to+\infty}{}_{\lambda}^{(q)}A_n\left(f\right)=f$, *pointwise and uniformly, and*

(4)

$$\left|{}_{\lambda}^{(q)}A_n\left(f,x\right)-f'\left(x\right){}_{\lambda}^{(q)}A_n\left(\sinh\left(\cdot-x\right),x\right)-2f''\left(x\right){}_{\lambda}^{(q)}A_n\left(\sinh^2\left(\frac{\cdot-x}{2}\right),x\right)-f\left(x\right)\right|$$
$$\leq\frac{\cosh\left(b-a\right)}{T_{\lambda}^{(q)}\left(1\right)}\left[\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''-f\right\|_{\infty}\left(b-a\right)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].$$

again here we achieve high speed of convergence at $n^{-3\alpha}$.

Proof. Similar to [5, pp. 197–198]. □

Next follows a mixed hyperbolic-trigonometric high order neural network approximation.

Theorem 4.3. *Let* $f\in C^4\left([a,b],\mathbb{C}\right)$, $0<\alpha<1$, $q,\lambda,n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. *Then*

(1)

$$\begin{aligned}
&\Big|{}_{\lambda}^{(q)}A_n(f,x)-f(x)-\frac{f'(x)}{2}\,{}_{\lambda}^{(q)}A_n\left(\left(\sinh(\cdot-x)+\sin(\cdot-x)\right),x\right)\\
&-\frac{f''(x)}{2}\,{}_{\lambda}^{(q)}A_n\left(\left(\cosh(\cdot-x)-\cos(\cdot-x)\right),x\right)-\frac{f^{(3)}(x)}{2}\,{}_{\lambda}^{(q)}A_n\left(\left(\sinh(\cdot-x)-\sin(\cdot-x)\right),x\right)\\
&-f^{(4)}(x)\,{}_{\lambda}^{(q)}A_n\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)\Big|\\
\leq\ &\frac{(\cosh(b-a)+1)}{2T_{\lambda}^{(q)}(1)}\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f^{(4)}-f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].
\end{aligned}$$

(2) *if* $f^{(i)}(x)=0$, $i=1,2,3,4$, *we get*

$$\begin{aligned}
\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right|\leq\ &\frac{(\cosh(b-a)+1)}{2T_{\lambda}^{(q)}(1)}\\
&\times\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f^{(4)}-f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].
\end{aligned}$$

in the last inequality observe the high speed of convergence at $n^{-3\alpha}$.

Proof. Similar to [5, pp. 207–208]. □

We continue with a general trigonometric result.

Theorem 4.4. *Let* $f\in C^4([a,b],\mathbb{C})$, $0<\alpha<1$, $q,\lambda,n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. *Let also* $\overline{\alpha},\overline{\beta}\in\mathbb{R}$ *with* $\overline{\alpha}\overline{\beta}\left(\overline{\alpha}^2-\overline{\beta}^2\right)\neq 0$. *Then*

(1)

$$\begin{aligned}
&\Bigg|{}_{\lambda}^{(q)}A_n(f,x)-f(x)-\frac{f'(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\,{}_{\lambda}^{(q)}A_n\left(\left(\overline{\beta}^3\sin(\overline{\alpha}(\cdot-x))-\overline{\alpha}^3\sin\left(\overline{\beta}(\cdot-x)\right)\right),x\right)\\
&-\frac{f''(x)}{\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\,{}_{\lambda}^{(q)}A_n\left(\left(\cos(\overline{\alpha}(\cdot-x))-\cos\left(\overline{\beta}(\cdot-x)\right)\right),x\right)\\
&-\frac{f'''(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\,{}_{\lambda}^{(q)}A_n\left(\left(\overline{\beta}\sin(\overline{\alpha}(\cdot-x))-\overline{\alpha}\sin\left(\overline{\beta}(\cdot-x)\right)\right),x\right)\\
&-\left(\frac{2f^{(4)}(x)+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''(x)}{\left(\overline{\alpha}\overline{\beta}\right)^2\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\right){}_{\lambda}^{(q)}A_n\left(\left(\overline{\beta}^2\sin^2\left(\frac{\overline{\alpha}(\cdot-x)}{2}\right)-\overline{\alpha}^2\sin^2\left(\frac{\overline{\beta}(\cdot-x)}{2}\right)\right),x\right)\Bigg|\\
\leq\ &\frac{1}{T_{\lambda}^{(q)}(1)\left|\overline{\beta}^2-\overline{\alpha}^2\right|}\left[\frac{\omega_1\left(\left(f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}\right.\\
&\left.+2\left\|f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].
\end{aligned}$$

(2) *if* $f^{(i)}(x)=0$, $i=1,2,3,4$, *we get*

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right|\leq\frac{1}{\left|\overline{\beta}^2-\overline{\alpha}^2\right|T_{\lambda}^{(q)}(1)}\left[\frac{\omega_1\left(\left(f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}\right.$$
$$\left.+2\left\|f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].$$

The high speed of convergence in (1) *and* (2) *is* $n^{-3\alpha}$.

Proof. Similar to [5, pp. 213–214]. □

We continue with a general hyperbolic result.

Theorem 4.5. *Let* $f\in C^4([a,b],\mathbb{C})$, $0<\alpha<1$, $q,\lambda,n\in\mathbb{N}: n^{1-\alpha}>2$, $x\in[a,b]$. *Let also* $\overline{\alpha},\overline{\beta}\in\mathbb{R}$ *with* $\overline{\alpha}\overline{\beta}\left(\overline{\alpha}^2-\overline{\beta}^2\right)\neq 0$. *Then*

(1)

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)-\frac{f'(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\,{}_{\lambda}^{(q)}A_n\left(\left(\overline{\beta}^3\sinh\left(\overline{\alpha}(\cdot-x)\right)-\overline{\alpha}^3\sinh\left(\overline{\beta}(\cdot-x)\right)\right),x\right)\right.$$
$$-\frac{f''(x)}{\overline{\beta}^2-\overline{\alpha}^2}\,{}_{\lambda}^{(q)}A_n\left(\left(\cosh\left(\overline{\beta}(\cdot-x)\right)-\cosh\left(\overline{\alpha}(\cdot-x)\right)\right),x\right)$$
$$-\frac{f'''(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\,{}_{\lambda}^{(q)}A_n\left(\left(\overline{\alpha}\sinh\left(\overline{\beta}(\cdot-x)\right)-\overline{\beta}\sinh\left(\overline{\alpha}(\cdot-x)\right)\right),x\right)$$
$$-\left(\frac{2(f^{(4)}(x)-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''(x))}{\left(\overline{\alpha}\overline{\beta}\right)^2\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\right)$$
$$\left.\times\,{}_{\lambda}^{(q)}A_n\left(\left(\overline{\alpha}^2\sinh^2\left(\frac{\overline{\beta}(\cdot-x)}{2}\right)-\overline{\beta}^2\sinh^2\left(\frac{\overline{\alpha}(\cdot-x)}{2}\right)\right),x\right)\right|$$
$$\leq\frac{\cosh(b-a)}{T_{\lambda}^{(q)}(1)\left|\overline{\beta}^2-\overline{\alpha}^2\right|}\left[\frac{\omega_1\left(\left(f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}\right.$$
$$\left.+2\left\|f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right],$$

(2) *if* $f^{(i)}(x)=0$, $i=1,2,3,4$, *we get*

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right|$$
$$\leq\frac{\cosh(b-a)}{\left|\overline{\beta}^2-\overline{\alpha}^2\right|T_{\lambda}^{(q)}(1)}\left[\frac{\omega_1\left(\left(f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}\right.$$
$$\left.+\,2\left\|f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right\|_{\infty}(b-a)^2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].$$

The high speed of convergence in (1) *and* (2) *is* $n^{-3\alpha}$.

Proof. Similar to [5, pp. 214–215]. □

Definition 4.1. ([4, p. 490]) *Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$. Let $f \in C(M,\mathbb{C})$, we define the first modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \le \delta}} |f(x)-f(y)|, \quad 0 < \delta \le diam(M).$$

If $\delta > diam(M)$, then

$$\omega_1(f,\delta) = \omega_1(f, diam(M)).$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$.

Lemma 4.1. ([4, p. 490]) *We have $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M,\mathbb{C})$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.*

In our results we use $p = \infty$.

Let now $f \in C^2\left(\prod_{i=1}^N [a_i,b_i], \mathbb{C}\right)$, $N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1,...,\alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1,...,N$, and $|\alpha| := \sum_{i=1}^N \alpha_i = l$, where $l = 0,1,2$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l.

We denote

$$\omega_{1,2}^{\max}(f_\alpha,h) := \max_{\alpha:|\alpha|=2} \omega_1(f_\alpha,h).$$

Call also

$$\|f_\alpha\|_{\infty,2}^{\max} := \max_{|\alpha|=2} \{\|f_\alpha\|_\infty\},$$

where $\|\cdot\|_\infty$ is the supremum norm.

Next we discuss the trigonometric multivariate multicomposite approximation by using the smoothness of f.

Theorem 4.6. *Let $f \in C^2\left(\prod_{i=1}^N [a_i,b_i], \mathbb{C}\right)$, $0 < \beta < 1$, $q,\lambda,n,N \in \mathbb{N}$, $n^{1-\beta} > 2$, $x, x_0 \in \left(\prod_{i=1}^N [a_i,b_i]\right)$, $a := (a_1,...,a_N)$, $b := (b_1,...,b_N)$. Then*

(*i*)

$$\left| {}_{\lambda}^{(q)}A_n(f,x) - f(x) - \left(\sum_{=1}^N \frac{\partial f(x)}{\partial x_i} {}_{\lambda}^{(q)}A_n((\cdot - x_i), x)\right)\sin(1) \right.$$

$$\left. - 4\left\{ \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+ \\ i=1,...,N,|\alpha|:=\sum_{i=1}^N \alpha_i=2}} f_\alpha(x) \left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right) {}_{\lambda}^{(q)}A_n\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x\right)\right\}\sin^2\left(\frac{1}{2}\right)\right|$$

$$\le \left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1}\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right)N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right)\right\}\right]\right.$$

$$\left. + \left[2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + 2\|f\|_\infty\right](1-\cos(1))\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\},$$

(*ii*) *assume that* $\frac{\partial f(x_0)}{\partial x_i}=0$, $i=1,...,N$, *and* $f_\alpha(x_0)=0$, $\alpha:|\alpha|=2$, *we have that*

$$\begin{aligned}
&\left|{}_{\lambda}^{(q)}A_n(f,x_0)-f(x_0)\right| \\
&\leq\left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1}\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]\right. \\
&\left.+\left[2\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+2\|f\|_\infty\right](1-\cos(1))\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\},
\end{aligned}$$

(*iii*)

$$\begin{aligned}
&\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right| \\
&\leq\left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1}\left\{\left\{\left\{\sum_{i=1}^N\left|\frac{\partial f(x)}{\partial x_i}\right|\left\{\frac{1}{n^\beta}+(b_i-a_i)\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\right\}\sin(1)\right.\right. \\
&+4\left\{\sum_{\alpha:|\alpha|=2}|f_\alpha(x)|\left(\frac{1}{\prod_{i=1}^N\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^N(b_i-a_i)^{\alpha_i}\right)\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right]\right\}\sin^2\left(\frac{1}{2}\right)\Bigg\} \\
&+\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]\right. \\
&\left.\left.+\left[2\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+2\|f\|_\infty\right](1-\cos(1))\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\right\},
\end{aligned}$$

and

(*iv*)

$$\begin{aligned}
&\left\|{}_{\lambda}^{(q)}A_n(f)-f\right\|_\infty \\
&\leq\left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1}\left\{\left\{\left\{\sum_{i=1}^N\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\left\{\frac{1}{n^\beta}+(b_i-a_i)\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\right\}\sin(1)\right.\right. \\
&+4\left\{\sum_{\alpha:|\alpha|=2}\|f_\alpha\|_\infty\left(\frac{1}{\prod_{i=1}^N\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^N(b_i-a_i)^{\alpha_i}\right)\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right]\right\}\sin^2\left(\frac{1}{2}\right)\Bigg\} \\
&+\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]\right. \qquad (4.1) \\
&\left.\left.+\left[2\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+2\|f\|_\infty\right](1-\cos(1))\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\right\}=:{}_{\lambda}^{(q)}\xi_n(f).
\end{aligned}$$

We observe that ${}_{\lambda}^{(q)}A_n\to I$ *(unit operator), as* $n\to\infty$, *pointwise and uniformly.*

Proof. Similar to [5, pp. 232–234]. □

We continue with the related hyperbolic approximation.

Theorem 4.7. *Let* $f \in C^2\left(\prod_{i=1}^N [a_i,b_i], \mathbb{C}\right)$, $0 < \beta < 1$, $q,\lambda,n,N \in \mathbb{N}$, $n^{1-\beta} > 2$, $x, x_0 \in \left(\prod_{i=1}^N [a_i,b_i]\right)$, $a := (a_1,...,a_N)$, $b := (b_1,...,b_N)$. *Then*

(*i*)

$$\left| {}_{\lambda}^{(q)}A_n(f,x) - f(x) - \left(\sum_{=1}^{N} \frac{\partial f(x)}{\partial x_i} \; {}_{\lambda}^{(q)}A_n\left((\cdot - x_i), x\right)\right)\sinh(1) \right.$$
$$\left. - 4\left\{ \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+ \\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x) \left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right) \; {}_{\lambda}^{(q)}A_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right\}\sinh^2\left(\frac{1}{2}\right)\right|$$
$$\leq \left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1}\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right)N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right)\right\}\right]\right.$$
$$\left. + \left[\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + \|f\|_\infty\right]\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\},$$

(*ii*) *assume that* $\frac{\partial f(x_0)}{\partial x_i} = 0$, $i = 1,...,N$, *and* $f_\alpha(x_0) = 0$, $\alpha : |\alpha| = 2$, *we have that*

$$\left| {}_{\lambda}^{(q)}A_n(f,x_0) - f(x_0)\right|$$
$$\leq \left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1}(\cosh(1))\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right)N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right)\right\}\right]\right.$$
$$\left. + \left[\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + \|f\|_\infty\right]\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\},$$

(*iii*)

$$\left| {}_{\lambda}^{(q)}A_n(f,x) - f(x)\right|$$
$$\leq \left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1}\left\{\left\{\left\{\sum_{i=1}^{N}\left|\frac{\partial f(x)}{\partial x_i}\right|\left\{\frac{1}{n^\beta} + (b_i-a_i)\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\right\}\sinh(1)\right.\right.$$
$$+ 4\left\{\sum_{\alpha:|\alpha|=2}|f_\alpha(x)|\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}} + \left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right]\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg\}$$
$$+ \cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right)N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right)\right\}\right]\right.$$
$$\left.\left. + \left[\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + \|f\|_\infty\right]\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\right\},$$

and

(*iv*)

$$\begin{aligned}
&\left\| {}_{\lambda}^{(q)}A_n(f) - f \right\|_\infty \\
&\leq \left[\left(T_\lambda^{(q)}(1) \right)^N \right]^{-1} \left\{ \left\{ \left\{ \sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \left\{ \frac{1}{n^\beta} + (b_i - a_i) \left(1 - \widetilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \right\} \right\} \right\} \sinh(1) \right. \\
&+ 4 \left\{ \sum_{\alpha:|\alpha|=2} \|f_\alpha\|_\infty \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{2\beta}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \left(1 - \widetilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \right] \right\} \sinh^2\left(\frac{1}{2} \right) \Bigg\} \\
&+ \cosh(1) \left\{ \left[\frac{1}{2} \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta} \right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta} \right) \right\} \right] \right. \qquad (4.2) \\
&\left. \left. + \left[\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + \|f\|_\infty \right] \left(1 - \widetilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \right\} \right\} =: {}_{\lambda}^{(q)}\psi_n(f).
\end{aligned}$$

We observe that ${}_{\lambda}^{(q)}A_n \to I$ *(unit operator), as* $n \to \infty$*, pointwise and uniformly.*

Proof. Similar to [5, pp. 241–243]. □

We make

Remark 4.1. *By (3.4) we get that* $\left\| {}_{\lambda}^{(q)}A_n(f) \right\|_\infty \leq \|f\|_\infty < \infty$*, and* ${}_{\lambda}^{(q)}A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], \mathbb{C} \right)$*, given that* $f \in C\left(\prod_{i=1}^N [a_i, b_i], \mathbb{C} \right)$*. Clearly then*

$$\left\| {}_{\lambda}^{(q)}A_n^2(f) \right\|_\infty = \left\| {}_{\lambda}^{(q)}A_n\left({}_{\lambda}^{(q)}A_n(f) \right) \right\|_\infty \leq \left\| {}_{\lambda}^{(q)}A_n(f) \right\|_\infty \leq \|f\|_\infty,$$

etc. Therefore we get

$$\left\| {}_{\lambda}^{(q)}A_n^k(f) \right\|_\infty \leq \|f\|_\infty, \quad \forall\ k \in \mathbb{N},$$

the contraction property. Also we see that

$$\left\| {}_{\lambda}^{(q)}A_n^k(f) \right\|_\infty \leq \left\| {}_{\lambda}^{(q)}A_n^{k-1}(f) \right\|_\infty \leq ... \leq \left\| {}_{\lambda}^{(q)}A_n(f) \right\|_\infty \leq \|f\|_\infty.$$

Also ${}_{\lambda}^{(q)}A_n(1) = 1$, ${}_{\lambda}^{(q)}A_n^k(1) = 1$, $\forall\ k \in \mathbb{N}$*. Following 18.14, pp. 401-402, of [2], similarly we obtain that*

$$\left\| {}_{\lambda}^{(q)}A_n^r f - f \right\|_\infty \leq r \left\| {}_{\lambda}^{(q)}A_n(f) - f \right\|_\infty, \quad r \in \mathbb{N}.$$

We give

Theorem 4.8. *All as in Theorems 4.6, 4.7. Then*

(*i*)

$$\left\| {}_{\lambda}^{(q)}A_n^r f - f \right\|_\infty \leq r\ {}_{\lambda}^{(q)}\xi_n(f),$$

where ${}_{\lambda}^{(q)}\xi_n(f)$ *as in (4.1).*

(*ii*)

$$\left\| {}_{\lambda}^{(q)}A_n^r f - f \right\|_\infty \leq r\ {}_{\lambda}^{(q)}\psi_n(f),$$

where ${}_{\lambda}^{(q)}\psi_n(f)$ *as in (4.2).*

So that the speed of convergence to the unit operator of ${}_{\lambda}^{(q)}A_n^r$ *is not worse than of* ${}_{\lambda}^{(q)}A_n$*, see also [2].*

5 Univariate and Multivariate Multicomposite Fuzzy Approximation Results

We need

Definition 5.1. *Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1,...,N$; $q, \lambda \in \mathbb{N}$. We define the fuzzy multivariate multicomposite averaged positive linear neural network operators $(x := (x_1,...,x_N) \in \left(\prod_{i=1}^{N}[a_i,b_i]\right))$:*

$$
\begin{aligned}
&{}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x_1,...,x_N) := {}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x) \\
&:= \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \widehat{Z}_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widehat{Z}_{\lambda}^{(q)}(nx-k)} \\
&= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor *} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor *} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right) \odot \left(\prod_{i=1}^{N} T_{\lambda}^{(q)}(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} T_{\lambda}^{(q)}(nx_i-k_i)\right)},
\end{aligned}
$$

In the univariate case we define

Definition 5.2. *Let $f \in C_{\mathcal{F}}([a,b])$, and $q, \lambda, n \in \mathbb{N} : \lceil na \rfloor \leq \lfloor nb \rceil$. We define the fuzzy univariate multicomposite averaged positive linear neural network operators $(x \in [a,b])$*

$$
{}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot T_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k)}.
$$

About fuzzy positive linear operators see [1, p. 100]. By [4, pp. 589, 620], we see that in both case univariate and multivariate we have that

$$
\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x)\right)_{\pm}^{(r)} = {}_{\lambda}^{(q)}A_n\left(f_{\pm}^{(r)},x\right),
$$

$\forall\, r \in [0,1]$, $\forall\, x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$ or $\forall\, x \in [a,b]$, respectively.

Furthermore in either case holds:

$$
D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x), f(x)\right) = \sup_{r\in[0,1]} \max\left\{\left|\left({}_{\lambda}^{(q)}A_n\left(f_{-}^{(r)},x\right)\right) - f_{-}^{(r)}(x)\right|, \left|{}_{\lambda}^{(q)}A_n\left(f_{+}^{(r)},x\right) - f_{+}^{(r)}(x)\right|\right\}.
$$

We need

Proposition 5.1. *Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, then ${}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f) \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, true also in univariate case.*

Proof. Let $(x_m)_{m\in\mathbb{N}}$, $x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$ such that $x_m \to x_0$, as $m \to +\infty$. We have that

$$
\begin{aligned}
&D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x_m),{}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x_0)\right)\\
&= D\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{k}{n}\right)\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx_m-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx_m-k)},\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{k}{n}\right)\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx_0-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx_0-k)}\right)\\
&\overset{\text{(by [1, p. 5])}}{\leq}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} D\left(f\left(\frac{k}{n}\right)\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx_m-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx_m-k)},f\left(\frac{k}{n}\right)\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx_0-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx_0-k)}\right)
\end{aligned}
$$

(next by [1, Lemma 5.3, p. 76])

$$
\leq\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|\frac{\widehat{Z}_{\lambda}^{(q)}(nx_m-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx_m-k)}-\frac{\widehat{Z}_{\lambda}^{(q)}(nx_0-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx_0-k)}\right|F\left(f\left(\frac{k}{n}\right),\widetilde{o}\right)\to 0,
$$

as $m\to+\infty$, also notice f is fuzzy bounded (by [1, p. 161]). The claim is proved. □

We present the following:

Theorem 5.1. *Let $f\in C_{\mathcal{F}}^{2}([a,b])$, $0<\alpha<1$, $q,\lambda,N,n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. Then*

(1)

$$
\begin{aligned}
&D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x),f(x)\right)\\
&\leq\frac{1}{T_{\lambda}^{(q)}(1)}\left[D\left(f'(x),\widetilde{o}\right)\left(\frac{1}{n^{\alpha}}+(b-a)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.\\
&+\frac{D\left(f''(x),\widetilde{o}\right)}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\\
&\left.+\left(\frac{\omega_1^{(\mathcal{F})}\left(f''\oplus f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+D^{*}\left(f''\oplus f,\widetilde{o}\right)(b-a)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right],
\end{aligned}
$$

(2) *if $f'(x)=f''(x)=\widetilde{o}$, we obtain*

$$
\begin{aligned}
&D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f,x),f(x)\right)\\
&\leq\frac{1}{T_{\lambda}^{(q)}(1)}\left(\frac{\omega_1^{(\mathcal{F})}\left(f''\oplus f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+D^{*}\left(f''\oplus f,\widetilde{o}\right)(b-a)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right),
\end{aligned}
$$

notice here the high rate of fuzzy convergence at $n^{-3\alpha}$,

(3) *furthermore we derive*

$$
\begin{aligned}
&D^{*}\left({}_{\lambda}^{(q)}A_{n}^{(\mathcal{F})}\left(f\right),f\right)\\
&\leq\frac{1}{T_{\lambda}^{(q)}\left(1\right)}\left[D\left(f',\widetilde{o}\right)\left(\frac{1}{n^{\alpha}}+\left(b-a\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.\\
&+\frac{D\left(f'',\widetilde{o}\right)}{2}\left(\frac{1}{n^{2\alpha}}+\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\\
&\left.+\left(\frac{\omega_{1}^{(\mathcal{F})}\left(f''\oplus f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+D^{*}\left(f''\oplus f,\widetilde{o}\right)\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right],
\end{aligned}
$$

i.e. $\lim\limits_{n\to+\infty_{\lambda}}{}^{(q)}A_{n}^{(\mathcal{F})}\left(f\right)\overset{(D,D^{*})}{=}f$, *fuzzy pointwise and uniformly.*

Proof. Here we apply Theorem 4.1 for $f_{\pm}^{(r)}\in C^{2}\left(\left[a,b\right],\mathbb{R}\right)$. We observe that

$$
\begin{aligned}
&\left|{}_{\lambda}^{(q)}A_{n}\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}\left(x\right)\right|\\
&\leq\frac{1}{T_{\lambda}^{(q)}\left(1\right)}\left[\left|\left(f_{\pm}^{(r)}\right)'\left(x\right)\right|\left(\frac{1}{n^{\alpha}}+\left(b-a\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.\\
&+\frac{\left|\left(f_{\pm}^{(r)}\right)''\left(x\right)\right|}{2}\left(\frac{1}{n^{2\alpha}}+\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\\
&+\frac{\left(\omega_{1}\left(\left(f_{\pm}^{(r)}\right)''+f_{\pm}^{(r)},\frac{1}{n^{\alpha}}\right)\right)}{2n^{2\alpha}}\\
&\left.+\left\|\left(f_{\pm}^{(r)}\right)''+f_{\pm}^{(r)}\right\|_{\infty}\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{(by (2.4), (2.5))}}{=}\frac{1}{T_{\lambda}^{(q)}\left(1\right)}\left[\left|\left(f'\right)_{\pm}^{(r)}\left(x\right)\right|\left(\frac{1}{n^{\alpha}}+\left(b-a\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.\\
&+\frac{\left|\left(f''\right)_{\pm}^{(r)}\left(x\right)\right|}{2}\left(\frac{1}{n^{2\alpha}}+\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\\
&+\frac{\left(\omega_{1}\left(\left(f''\right)_{\pm}^{(r)}+f_{\pm}^{(r)},\frac{1}{n^{\alpha}}\right)\right)}{2n^{2\alpha}}\\
&\left.+\left\|\left(f''\right)_{\pm}^{(r)}+f_{\pm}^{(r)}\right\|_{\infty}\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{(by (2.1), (2.2), (2.3), (5.1))}}{\leq}\frac{1}{T_{\lambda}^{(q)}\left(1\right)}\left[D\left(f'\left(x\right),\widetilde{o}\right)\left(\frac{1}{n^{\alpha}}+\left(b-a\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\right.\\
&+\frac{D\left(f''\left(x\right),\widetilde{o}\right)}{2}\left(\frac{1}{n^{2\alpha}}+\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right)\\
&\left.+\frac{\left(\omega_{1}\left(f''\oplus f,\frac{1}{n^{\alpha}}\right)\right)}{2n^{2\alpha}}+D^{*}\left(f''\oplus f,\widetilde{o}\right)\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].
\end{aligned}
$$

Above notice that

$$\left(f''\right)_{\pm}^{(r)}+f_{\pm}^{(r)}=\left(f''\oplus f\right)_{\pm}^{(r)}. \tag{5.1}$$

□

We continue with a fuzzy trigonometric result.

Theorem 5.2. *Let $f\in C_{\mathcal{F}}^{4}\left([a,b]\right)$, $0<\alpha<1$, $q,\lambda,n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. Let also $\overline{\alpha},\overline{\beta}\in\mathbb{R}$ with $\overline{\alpha}\overline{\beta}\left(\overline{\alpha}^{2}-\overline{\beta}^{2}\right)\neq 0$. We assume that $f^{(i)}\left(x\right)=\widetilde{o}$, $i=1,2,3,4$. Then*

$$\begin{aligned}
&D\left({}_{\lambda}^{(q)}A_{n}^{(\mathcal{F})}\left(f,x\right),f\left(x\right)\right)\\
&\leq\frac{1}{\left|\overline{\beta}^{2}-\overline{\alpha}^{2}\right|T_{\lambda}^{(q)}\left(1\right)}\left[\frac{\omega_{1}^{(\mathcal{F})}\left(\left(f^{(4)}\oplus\left(\overline{\alpha}^{2}+\overline{\beta}^{2}\right)\odot f''\oplus\overline{\alpha}^{2}\overline{\beta}^{2}\odot f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}\right.\\
&\left.+2D^{*}\left(\left(f^{(4)}\oplus\left(\overline{\alpha}^{2}+\overline{\beta}^{2}\right)\odot f''\oplus\overline{\alpha}^{2}\overline{\beta}^{2}\odot f\right),\widetilde{o}\right)\left(b-a\right)^{2}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\right].
\end{aligned}\tag{5.2}$$

The high speed of fuzzy convergence in (5.2) is $n^{-3\alpha}$.

Proof. As long and similar to Theorem 5.1 is omitted, based on Theorem 4.4. □

Next come multidimensional multiactivated results:

First a trigonometric one.

Theorem 5.3. *Let $f\in C_{\mathcal{F}}^{2}\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$, $0<\beta<1$, $q,\lambda,n,N\in\mathbb{N}$, $n^{1-\beta}>2$, $x,x_{0}\in\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$, $a:=(a_{1},...,a_{N})$, $b:=(b_{1},...,b_{N})$. Then*

(1) *if $\frac{\partial f(x_{0})}{\partial x_{i}}=\widetilde{o}$, $i=1,...,N$, and $f_{\alpha}\left(x_{0}\right)=\widetilde{o}$, $\alpha:|\alpha|=2$, we have*

$$\begin{aligned}
&D\left({}_{\lambda}^{(q)}A_{n}^{(\mathcal{F})}\left(f,x_{0}\right),f\left(x_{0}\right)\right)\\
&\leq\left[\left(T_{\lambda}^{(q)}\left(1\right)\right)^{N}\right]^{-1}\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{(\mathcal{F})}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right.\\
&\left.+\left[2\left\|b-a\right\|_{\infty}^{2}\left(D^{*}\left(f_{\alpha:|\alpha|=2},\widetilde{o}\right)\right)_{\max}N^{2}+2D^{*}\left(f,\widetilde{o}\right)\right]\left(1-\cos\left(1\right)\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\right\},
\end{aligned}$$

(2)

$$D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}\left(f,x\right),f\left(x\right)\right)\leq\left[\left(T_{\lambda}^{(q)}\left(1\right)\right)^N\right]^{-1}$$
$$\times\left\{\left\{\left\{\sum_{i=1}^{N}D\left(\frac{\partial f\left(x\right)}{\partial x_i},\widetilde{o}\right)\left\{\frac{1}{n^{\beta}}+\left(b_i-a_i\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\right\}\right\}\sin\left(1\right)$$
$$+4\left\{\sum_{\alpha:|\alpha|=2}D\left(f_{\alpha}\left(x\right),\widetilde{o}\right)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_i-a_i\right)^{\alpha_i}\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\right]\right\}\sin^2\left(\frac{1}{2}\right)\Bigg\}$$
$$+\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{(\mathcal{F})}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right.$$
$$\left.+\left[2\left\|b-a\right\|_{\infty}^2\left(D^*\left(f_{\alpha:|\alpha|=2},\widetilde{o}\right)\right)_{\max}N^2+2D^*\left(f,\widetilde{o}\right)\right]\right.$$
$$\left.\times\left(1-\cos\left(1\right)\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\},$$

and

(3)

$$D^*\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}\left(f\right),f\right)\leq\left[\left(T_{\lambda}^{(q)}\left(1\right)\right)^N\right]^{-1}$$
$$\times\left\{\left\{\left\{\sum_{i=1}^{N}D^*\left(\frac{\partial f}{\partial x_i},\widetilde{o}\right)\left\{\frac{1}{n^{\beta}}+\left(b_i-a_i\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\right\}\right\}\sin\left(1\right)$$
$$+4\left\{\sum_{\alpha:|\alpha|=2}D\left(f_{\alpha},\widetilde{o}\right)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_i-a_i\right)^{\alpha_i}\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\right]\right\}\sin^2\left(\frac{1}{2}\right)\Bigg\}$$
$$+\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{(\mathcal{F})}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right.$$
$$\left.+\left[2\left\|b-a\right\|_{\infty}^2\left(D^*\left(f_{\alpha:|\alpha|=2},\widetilde{o}\right)\right)_{\max}N^2+2D^*\left(f,\widetilde{o}\right)\right]\right.$$
$$\left.\times\left(1-\cos\left(1\right)\right)\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}\}.$$

We see that ${}_{\lambda}^{(q)}A_n^{(\mathcal{F})}\overset{(D,D*)}{\to}I$ *(unit operator), as* $n\to\infty$*, fuzzy pointwise and uniformly.*

Proof. As long and similar to Theorem 5.1 is omitted. It is based on Theorem 4.6. □

Next comes the corresponding hyperbolic result:

Theorem 5.4. *Let* $f\in C_{\mathcal{F}}^2\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $0<\beta<1$, $q,\lambda,n,N\in\mathbb{N}$, $n^{1-\beta}>2$, $x,x_0\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $a:=(a_1,...,a_N)$, $b:=(b_1,...,b_N)$. *Then*

(1) *if* $\frac{\partial f(x_0)}{\partial x_i} = \widetilde{o}$, $i = 1, ..., N$, *and* $f_\alpha(x_0) = \widetilde{o}$, $\alpha : |\alpha| = 2$, *we have*

$$\begin{aligned} & D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f, x_0), f(x_0)\right) \\ \leq & \left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1}(\cosh(1))\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{(\mathcal{F})}\left(f_\alpha, \frac{1}{n^\beta}\right)N^2}{n^{2\beta}} + \omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\beta}\right)\right\}\right]\right. \\ & \left. + \left[\|b-a\|_\infty^2 \left(D^*\left(f_{\alpha:|\alpha|=2}, \widetilde{o}\right)\right)_{\max} N^2 + D^*(f, \widetilde{o})\right]\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta} - 2\right)\right)\right\}, \end{aligned}$$

(2)

$$\begin{aligned} & D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f, x), f(x)\right) \leq \left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1} \\ & \times\left\{\left\{\left\{\sum_{i=1}^N D\left(\frac{\partial f(x)}{\partial x_i}, \widetilde{o}\right)\left\{\frac{1}{n^\beta} + (b_i - a_i)\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta} - 2\right)\right)\right\}\right\}\right\}\sinh(1)\right. \\ & + 4\left\{\sum_{\alpha:|\alpha|=2} D(f_\alpha(x), \widetilde{o})\left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)\left[\frac{1}{n^{2\beta}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i}\right)\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta} - 2\right)\right)\right]\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg\} \\ & + \cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{(\mathcal{F})}\left(f_\alpha, \frac{1}{n^\beta}\right)N^2}{n^{2\beta}} + \omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\beta}\right)\right\}\right]\right. \\ & \left.\left. + \left[\|b-a\|_\infty^2 \left(D^*\left(f_{\alpha:|\alpha|=2}, \widetilde{o}\right)\right)_{\max} N^2 + D^*(f, \widetilde{o})\right]\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta} - 2\right)\right)\right\}\right\}, \end{aligned}$$

and

(3)

$$\begin{aligned} & D^*\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f), f\right) \leq \left[\left(T_\lambda^{(q)}(1)\right)^N\right]^{-1} \\ & \times\left\{\left\{\left\{\sum_{i=1}^N D^*\left(\frac{\partial f}{\partial x_i}, \widetilde{o}\right)\left\{\frac{1}{n^\beta} + (b_i - a_i)\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta} - 2\right)\right)\right\}\right\}\right\}\sinh(1)\right. \\ & + 4\left\{\sum_{\alpha:|\alpha|=2} D(f_\alpha, \widetilde{o})\left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)\left[\frac{1}{n^{2\beta}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i}\right)\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta} - 2\right)\right)\right]\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg\} \\ & + \cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{(\mathcal{F})}\left(f_\alpha, \frac{1}{n^\beta}\right)N^2}{n^{2\beta}} + \omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\beta}\right)\right\}\right]\right. \\ & \left.\left. + \left[\|b-a\|_\infty^2 \left(D^*\left(f_{\alpha:|\alpha|=2}, \widetilde{o}\right)\right)_{\max} N^2 + D^*(f, \widetilde{o})\right]\left(1 - \widetilde{G}_\lambda^{(q)}\left(n^{1-\beta} - 2\right)\right)\right\}\right\}. \end{aligned}$$

We see that ${}_{\lambda}^{(q)}A_n^{(\mathcal{F})} \overset{(D,D^*)}{\to} I$ *(unit operator), as* $n \to \infty$, *fuzzy pointwise and uniformly.*

Proof. As long and similar to Theorem 5.1 is omitted. It is based on Theorem 4.7. □

We finish with:

Remark 5.1. *By Proposition 5.1 if* $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, *then* ${}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f) \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ *and all iterates* $\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f)\right)^{(j)} \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $j=1,2,3,....$ *We notice that* $(x \in \prod_{i=1}^{N}[a_i,b_i])$

$$D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f)(x),\widetilde{o}\right) = D\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{k}{n}\right)\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx-k)},\widetilde{o}\right)$$

$$= D\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{k}{n}\right)\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx-k)},\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *}\widetilde{o}\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx-k)}\right)$$

(*by* [1, p. 5])

$$\leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} D\left(f\left(\frac{k}{n}\right)\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx-k)},\widetilde{o}\odot\frac{\widehat{Z}_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx-k)}\right)$$

$$= \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\widehat{Z}_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widehat{Z}_{\lambda}^{(q)}(nx-k)} D\left(f\left(\frac{k}{n}\right),\widetilde{o}\right) \leq D^*(f,\widetilde{o}),$$

that is

$$D\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}f,\widetilde{o}\right) \leq D^*(f,\widetilde{o}),$$

and likewise it holds

$$D^*\left(\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f)\right)^{(2)},\widetilde{o}\right) \leq D^*\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f),\widetilde{o}\right) \leq D^*(f,\widetilde{o}).$$

Inductively, we get that

$$\begin{aligned} D^*\left(\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f)\right)^{(m)},\widetilde{o}\right) &\leq D^*\left(\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f)\right)^{(m-1)},\widetilde{o}\right) \\ &\leq \cdots \leq D^*\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f),\widetilde{o}\right) \leq D^*(f,\widetilde{o}), \end{aligned}$$

$\forall m \in \mathbb{N}$. *That is*

$$D^*\left(\left({}_{\lambda}^{(q)}A_n^{(\mathcal{F})}(f)\right)^{(m)},\widetilde{o}\right) \leq D^*(f,\widetilde{o}), \quad \forall\ m \in \mathbb{N},$$

proving the contraction property. All the above are true in the univariate case too.

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