



RESEARCH ARTICLE

Investigating the Quantitative Behavior of Kantorovich Fractional Operators in q -Numbers

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Abstract: This research introduces the Kantorovich-type fractional Baskakov operators defined via q -integers and investigates their approximation properties. We compute the moments and central moments of these operators and establish Korovkin-type approximation theorems along with estimates of their rate of convergence. The analysis is carried out using tools such as the Lipschitz class, Lipschitz-type maximal functions, Peetre's K -functional, and the first- and second-order moduli of smoothness. Furthermore, a Voronovskaya-type theorem is proved, and the weighted approximation properties of the operators are examined. Finally, a numerical comparison of the proposed operators is presented, supported by graphical illustrations.

Keywords: Modulus of continuity, rate of convergence, q -calculus, Voronovskaya approximation, weighted approximation.

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1 Introduction

Positive linear operators have been extensively studied because they are useful in many areas of mathematics, such as functional analysis, operator theory, probability theory, and applied mathematics. The systematic investigation of positive linear operators in approximation theory started with the innovative study of S. N. Bernstein in 1912 [1], which introduced polynomial operators to approximate continuous functions on the interval $[0, 1]$. Motivated by Bernstein's crucial work, numerous generalizations and extensions have been introduced [2–6].

The Kantorovich version of positive linear operators is important in approximation theory because it makes classical operators work with integrable functions as well as continuous functions. The Kantorovich modification substitutes pointwise evaluations of the function with appropriate integral means over small subintervals, in comparison with classical positive linear operators. They also give a natural structure for looking into quantitative estimates like rates of convergence, moment estimates, Voronovskaya-type asymptotic formulas, and Grüss–Voronovskaya inequalities. Because of these benefits, Kantorovich versions have been thoroughly examined for different types of operators,

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such as Bernstein-Kantorovich operators [7], Szász-Mirakjan-Kantorovich operators [8], Baskakov-Kantorovich operators [9], Lupaş-Kantorovich operators [2]. Recently Kantorovich operators based on Hermite polynomials have been discussed in [10].

Fractional integrals are important for defining and analyzing generalized positive linear operators, resulting in enhanced approximation outcomes for functions exhibiting limited regularity. Fractional integrals lead to more precise estimates for rates of convergence, moment inequalities, and asymptotic expressions, including Voronovskaya-type theorems. Also, operators based on fractional integrals naturally expand approximation theory to larger function spaces, which makes the theoretical basis stronger and makes approximation methods more useful for problems in applied mathematics, mathematical physics, and engineering. In recent years, numerous researchers have developed novel Kantorovich-type operators utilizing the Riemann–Liouville fractional integral (refer to [11–15]). In this work, we use specific notations and fundamental concepts of fractional q -calculus, adapted from [16].

The theory of q -type operators has developed into a significant extension of classical approximation theory by integrating the approaches of q -calculus, which offers an inherent framework for the separation and deformation of classical operators. These operators provide enhanced flexibility in managing the approximation process through the introduction of the parameter q , facilitating a smooth transition between discrete and continuous contexts as $q \rightarrow 1^-$. Lupaş ([17]) proposed the q -Bernstein polynomials in 1987. Afterward, Phillips [18] introduced additional q -analogs of the Bernstein polynomials for $f \in C([0, 1])$. In 2009, Aral and Gupta [19] described the q -analogue of Baskakov operators as

$$V_{n,q}(f; x) = \sum_{k=0}^{\infty} v_{n,k}(q; x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right), \quad (1.1)$$

for functions for which the series converges, where $q \in (0, 1]$, $x \in [0, \infty)$ and

$$v_{n,k}(q; x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}}. \quad (1.2)$$

The Kantorovich variants of the q -Bernstein and q -Baskakov operators were examined in ([20, 21]). There has been significant work on asymptotic results, statistical convergence, and integral variants of q -Bernstein-type operators ([22–24]).

For $q \in (0, 1)$, $n \in \mathbb{N} \cup \{0\}$ and $x_1, x_2 \in \mathbb{R}$, q -analogue of the power function is defined as

$$(x_1 - x_2)_q^{(n)} = \begin{cases} \prod_{j=0}^{n-1} (x_1 - x_2 q^j), & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

If $n \in \mathbb{R}$ then

$$(x_1 - x_2)_q^{(n)} = x_1^n \prod_{j=0}^{\infty} \frac{x_1 - x_2 q^j}{x_1 - x_2 q^{n+j}}, \quad x_1 \neq x_2 q^{n+j}, \text{ for } j = 0, 1, 2, \dots$$

For $x_2 = 0$, $(x_1)_q^{(n)} = x_1^n$. For further discussion, we recall the definition of the Riemann–Liouville fractional q -integral.

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - q\mu)_q^{(\alpha-1)} f(\mu) d_q \mu, \quad \alpha \in (0, \infty)$$

and $(I_q^0 f)(\mu) = f(\mu)$.

We consider the weighted function spaces associated with the weight function $\rho(x) = 1 + x^2$, $x \in [0, \infty)$. We define

$$C_\rho([0, \infty)) = \{f \in C([0, \infty)) : \|f\|_\rho < \infty\},$$

where the weighted norm is given by

$$\|f\|_\rho = \sup_{x \geq 0} \frac{|f(x)|}{\rho(x)} = \sup_{x \geq 0} \frac{|f(x)|}{1 + x^2}.$$

Furthermore, we consider the subspace

$$C_\rho^*([0, \infty)) = \left\{ f \in C_\rho([0, \infty)) : \lim_{x \rightarrow \infty} \frac{f(x)}{\rho(x)} \text{ exists and is finite} \right\}.$$

In [25], Kara introduced the fractional q -integral of Riemann-Liouville-type Szász-Mirakjan-Kantorovich operators. Motivated by this for $\alpha \in (0, \infty)$, $q \in (0, 1)$ and $f \in C_\rho^*([0, \infty))$, we introduce the following q -Riemann-Liouville-Baskakov-Kantorovich (q -RLBaK) operators

$$V_{n,q}^\alpha(f; x) = \frac{\Gamma_q(\alpha + 1)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} v_{n,k}(q; x) \int_0^1 (1 - q\mu)_q^{(\alpha-1)} f\left(\frac{q(q^{-k}[k]_q + \mu)}{[n]_q}\right) d_q\mu, \quad (1.3)$$

where $v_{n,k}(q; x)$ is defined as in (1.2). The operator $V_{n,q}^\alpha(f)$ is well defined for every $f \in C_\rho^*([0, \infty))$, as established in Lemma 2.5. For different values of q and α , q -RLBaK operators (1.3) reduce to various types of operators (see Table 1).

Table 1: For different values of q and α , q -Riemann-Liouville-Kantorovich-type operators

q -RLBaK operators	$\alpha = 1, q \rightarrow 1^-$	Baskakov-Kantorovich ([9])
	$q \rightarrow 1^-$	Riemann-Liouville-Baskakov-Kantorovich [12]

The operators defined in (1.3) generalize the classical Baskakov-Kantorovich operators by combining q -calculus with Riemann-Liouville fractional integration. The parameters $q \in (0, 1)$ and $\alpha > 0$ provide additional flexibility within this operator family. For $\alpha = 1$, the classical Baskakov-Kantorovich operators are recovered in the limit $q \rightarrow 1^-$. The present study establishes moment formulas, quantitative approximation estimates, weighted convergence, and a Voronovskaya-type asymptotic formula for the proposed operators.

2 Moment and Central Moment Estimates

Let $C_B([0, \infty))$ be the space of all real valued continuous and bounded functions on $[0, \infty)$ and $UC_B([0, \infty))$ be the space of all real valued uniformly continuous and bounded functions on $[0, \infty)$. We will use the following symbols throughout the paper. For $m = 0, 1, 2, \dots$, $e_m(\mu) = \mu^m$ and $\psi_{x,m}(\mu) = (\mu - x)^m$ for any $\mu, x \in [0, \infty)$.

Lemma 2.1. ([19, 26]) *Let $q \in (0, 1)$ and $n \in \mathbb{N}$, then for a given $x \in [0, \infty)$, moments of the q -Baskakov operators (1.1) are given by*

$$V_{n,q}(e_0; x) = 1, \quad V_{n,q}(e_1; x) = x,$$

$$V_{n,q}(e_2; x) = \frac{x}{[n]_q} + \frac{[n+1]_q x^2}{q[n]_q},$$

$$\begin{aligned}
V_{n,q}(e_3; x) &= \frac{x}{[n]_q^2} + \frac{(1+2q)[n+1]_q}{q^2[n]_q^2} x^2 + \frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} x^3, \\
V_{n,q}(e_4; x) &= \frac{x}{[n]_q^3} + \frac{(1+3q+3q^2)[n+1]_q}{q^3[n]_q^3} x^2 + \frac{(1+2q+3q^2)[n+1]_q[n+2]_q}{q^5[n]_q^3} x^3 \\
&\quad + \frac{[n+1]_q[n+2]_q[n+3]_q}{q^6[n]_q^3} x^4.
\end{aligned}$$

Lemma 2.2. Let $\alpha \in (0, \infty)$, $q \in (0, 1)$, $n \in \mathbb{N}$ and $f \in C_B([0, \infty))$, then for a given $x \in [0, \infty)$, we have $|V_{n,q}^\alpha(f; x)| \leq \|f\|$, where the symbol $\|\cdot\|$ denotes the usual supremum norm on $[0, \infty)$.

Proof. Using the integral inequality, we get

$$\begin{aligned}
|V_{n,q}^\alpha(f; x)| &\leq \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} v_{n,k}(q; x) \int_0^1 (1-q\mu)_q^{(\alpha-1)} \left| f\left(\frac{q(q^{-k}[k]_q + \mu)}{[n]_q}\right) \right| d_q\mu \\
&\leq \|f\| \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} v_{n,k}(q; x) \int_0^1 (1-q\mu)_q^{(\alpha-1)} d_q\mu = \|f\|,
\end{aligned}$$

which completes the proof of Lemma 2.2. $\square$

Lemma 2.3. Let $\alpha \in (0, \infty)$, $q \in (0, 1)$ and $n \in \mathbb{N}$, then for a given $x \in [0, \infty)$, we have the following

$$V_{n,q}^\alpha(e_m; x) = \frac{q^m \Gamma_q(\alpha+1)}{[n]_q^m} \sum_{j=0}^m \binom{m}{j} \frac{\Gamma_q(m-j+1)}{q^j \Gamma_q(m-j+1+\alpha)} [n]_q^j V_{n,q}(e_j; x),$$

where $V_{n,q}$ is the q -analogue of Baskakov operators, defined in (1.1).

Proof. We immediately get

$$\begin{aligned}
V_{n,q}^\alpha(e_m; x) &= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} v_{n,k}(q; x) \int_0^1 (1-q\mu)_q^{(\alpha-1)} \left(\frac{q(q^{-k}[k]_q + \mu)}{[n]_q} \right)^m d_q\mu \\
&= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} \frac{q^m}{[n]_q^m} \sum_{k=0}^{\infty} v_{n,k}(q; x) \sum_{j=0}^m \binom{m}{j} q^{(-k)j} [k]_q^j \int_0^1 (1-q\mu)_q^{(\alpha-1)} \mu^{m-j} d_q\mu \\
&= \frac{q^m \Gamma_q(\alpha+1)}{[n]_q^m} \sum_{j=0}^m \binom{m}{j} \frac{\Gamma_q(m-j+1)}{q^j \Gamma_q(m-j+1+\alpha)} [n]_q^j V_{n,q}(e_j; x),
\end{aligned}$$

which is the desired result. $\square$

Lemma 2.4. Let $\alpha \in (0, \infty)$, $q \in (0, 1)$ and $n \in \mathbb{N}$, then for a given $x \in [0, \infty)$, we achieve

$$\begin{aligned}
V_{n,q}^\alpha(e_0; x) &= 1, \quad V_{n,q}^\alpha(e_1; x) = x + \frac{q}{[n]_q[\alpha+1]_q}, \\
V_{n,q}^\alpha(e_2; x) &= \left(\frac{[n+1]_q}{q[n]_q} \right) x^2 + \left(\frac{2q + [\alpha+1]_q}{[n]_q[\alpha+1]_q} \right) x + \frac{q^2[2]_q}{[n]_q^2[\alpha+2]_q[\alpha+1]_q}, \\
V_{n,q}^\alpha(e_3; x) &= \frac{q^3[3]_q[2]_q}{[n]_q^3[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \left(\frac{3[2]_q q^2}{[\alpha+2]_q[\alpha+1]_q} + 1 + \frac{3q}{[\alpha+1]_q} \right) \frac{x}{[n]_q^2} \\
&\quad + \left(\frac{3[n+1]_q}{[\alpha+1]_q} + \frac{(1+2q)[n+1]_q}{q^2} \right) \frac{x^2}{[n]_q^2} + \frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} x^3,
\end{aligned}$$

$$\begin{aligned}
V_{n,q}^\alpha(e_4; x) &= \frac{q^4[4]_q[3]_q[2]_q}{[n]_q^4[\alpha+4]_q[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} \\
&+ \left(\frac{4[3]_q[2]_q q^3}{[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \frac{6[2]_q q^2}{[\alpha+2]_q[\alpha+1]_q} + \frac{4q}{[\alpha+1]_q} + 1 \right) \frac{x}{[n]_q^3} \\
&+ \left(\frac{6[2]_q q}{[\alpha+2]_q[\alpha+1]_q} + \frac{4(1+2q)}{q[\alpha+1]_q} + \frac{(1+3q+3q^2)}{q^3} \right) \frac{[n+1]_q x^2}{[n]_q^3} \\
&+ \left(\frac{4}{q^2[\alpha+1]_q} + \frac{(1+2q+3q^2)}{q^5} \right) \frac{[n+1]_q[n+2]_q x^3}{[n]_q^3} \\
&+ \frac{[n+1]_q[n+2]_q[n+3]_q}{q^6[n]_q^3} x^4.
\end{aligned}$$

Proof. Using (1.3), we get

$$V_{n,q}^\alpha(e_0; x) = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} v_{n,k}(q; x) B_q(\alpha, 1) = 1.$$

We also observe that

$$V_{n,q}^\alpha(e_1; x) = x + \frac{q}{[n]_q[\alpha+1]_q}.$$

For e_2 , we obtain

$$\begin{aligned}
V_{n,q}^\alpha(e_2; x) &= \frac{q^2 \Gamma_q(\alpha+1)}{[n]_q^2} \sum_{j=0}^2 \binom{2}{j} \frac{\Gamma_q(3-j)}{q^j \Gamma_q(3-j+\alpha)} [n]_q^j V_{n,q}(e_j; x) \\
&= \left(\frac{[n+1]_q}{q[n]_q} \right) x^2 + \left(\frac{2q + [\alpha+1]_q}{[n]_q[\alpha+1]_q} \right) x + \frac{q^2[2]_q}{[n]_q^2[\alpha+2]_q[\alpha+1]_q},
\end{aligned}$$

and for e_3 , we achieve

$$\begin{aligned}
V_{n,q}^\alpha(e_3; x) &= \frac{q^3 \Gamma_q(\alpha+1)}{[n]_q^3} \sum_{j=0}^3 \binom{3}{j} \frac{\Gamma_q(4-j)}{q^j \Gamma_q(4-j+\alpha)} [n]_q^j V_{n,q}(e_j; x) \\
&= \frac{q^3[3]_q[2]_q}{[n]_q^3[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \frac{3[2]_q q^2 x}{[n]_q^2[\alpha+2]_q[\alpha+1]_q} \\
&+ \frac{3q}{[n]_q[\alpha+1]_q} \left(\frac{x}{[n]_q} + \frac{[n+1]_q x^2}{q[n]_q} \right) \\
&+ \frac{x}{[n]_q^2} + \frac{(1+2q)[n+1]_q}{q^2[n]_q^2} x^2 + \frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} x^3 \\
&= \frac{q^3[3]_q[2]_q}{[n]_q^3[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \left(\frac{3[2]_q q^2}{[\alpha+2]_q[\alpha+1]_q} + 1 + \frac{3q}{[\alpha+1]_q} \right) \frac{x}{[n]_q^2} \\
&+ \left(\frac{3[n+1]_q}{[\alpha+1]_q} + \frac{(1+2q)[n+1]_q}{q^2} \right) \frac{x^2}{[n]_q^2} + \frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} x^3.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
V_{n,q}^\alpha(e_4; x) &= \frac{q^4 \Gamma_q(\alpha+1)}{[n]_q^4} \sum_{j=0}^4 \binom{4}{j} \frac{\Gamma_q(5-j)}{q^j \Gamma_q(5-j+\alpha)} [n]_q^j V_{n,q}(e_j; x) \\
&= \frac{q^4 [4]_q [3]_q [2]_q}{[n]_q^4 [\alpha+4]_q [\alpha+3]_q [\alpha+2]_q [\alpha+1]_q} + \frac{4 [3]_q [2]_q q^3 x}{[n]_q^3 [\alpha+3]_q [\alpha+2]_q [\alpha+1]_q} \\
&\quad + \frac{6 [2]_q q^2}{[n]_q^2 [\alpha+2]_q [\alpha+1]_q} \left(\frac{x}{[n]_q} + \frac{[n+1]_q x^2}{q [n]_q} \right) \\
&\quad + \frac{4q}{[n]_q [\alpha+1]_q} \left(\frac{x}{[n]_q^2} + \frac{(1+2q)[n+1]_q}{q^2 [n]_q^2} x^2 + \frac{[n+1]_q [n+2]_q}{q^3 [n]_q^2} x^3 \right) + \frac{x}{[n]_q^3} \\
&\quad + \frac{(1+3q+3q^2)[n+1]_q}{q^3 [n]_q^3} x^2 + \frac{(1+2q+3q^2)[n+1]_q [n+2]_q}{q^5 [n]_q^3} x^3 \\
&\quad + \frac{[n+1]_q [n+2]_q [n+3]_q}{q^6 [n]_q^3} x^4.
\end{aligned}$$

Thus,

$$\begin{aligned}
V_{n,q}^\alpha(e_4; x) &= \frac{q^4 [4]_q [3]_q [2]_q}{[n]_q^4 [\alpha+4]_q [\alpha+3]_q [\alpha+2]_q [\alpha+1]_q} \\
&\quad + \left(\frac{6 [2]_q q}{[\alpha+2]_q [\alpha+1]_q} + \frac{4(1+2q)}{q [\alpha+1]_q} + \frac{(1+3q+3q^2)}{q^3} \right) \frac{[n+1]_q x^2}{[n]_q^3} \\
&\quad + \left(\frac{4 [3]_q [2]_q q^3}{[\alpha+3]_q [\alpha+2]_q [\alpha+1]_q} + \frac{6 [2]_q q^2}{[\alpha+2]_q [\alpha+1]_q} + \frac{4q}{[\alpha+1]_q} + 1 \right) \frac{x}{[n]_q^3} \\
&\quad + \left(\frac{4}{q^2 [\alpha+1]_q} + \frac{(1+2q+3q^2)}{q^5} \right) \frac{[n+1]_q [n+2]_q x^3}{[n]_q^3} \\
&\quad + \frac{[n+1]_q [n+2]_q [n+3]_q}{q^6 [n]_q^3} x^4.
\end{aligned}$$

This concludes the proof of Lemma 2.4. $\square$

Lemma 2.5. Let $\alpha \in (0, \infty)$, $q \in (0, 1)$, and $n \in \mathbb{N}$. Then the operator $V_{n,q}^\alpha$ is well defined for every $f \in C_\rho^*([0, \infty))$, where $\rho(\mu) = 1 + \mu^2$ for all $\mu \in [0, \infty)$.

Proof. Let $f \in C_\rho^*([0, \infty))$. By the definition of $C_\rho^*([0, \infty))$, there exists a constant $M_f > 0$ such that

$$|f(\mu)| \leq M_f \rho(\mu) = M_f (1 + \mu^2), \quad \mu \in [0, \infty).$$

Since $V_{n,q}^\alpha$ is a positive linear operator, we have

$$|V_{n,q}^\alpha(f; x)| \leq V_{n,q}^\alpha(|f|; x) \leq M_f V_{n,q}^\alpha(1 + \mu^2; x).$$

Using the linearity of operators $V_{n,q}^\alpha$, we get

$$|V_{n,q}^\alpha(f; x)| \leq M_f V_{n,q}^\alpha(1 + \mu^2; x) = M_f (V_{n,q}^\alpha(e_0; x) + V_{n,q}^\alpha(e_2; x)).$$

Using the moment estimates established in Lemma 2.4, we obtain

$$|V_{n,q}^\alpha(f; x)| \leq M_f \left(1 + \frac{[n+1]_q}{q [n]_q} x^2 + \frac{2q + [\alpha+1]_q}{[n]_q [\alpha+1]_q} x + \frac{q^2 [2]_q}{[n]_q^2 [\alpha+2]_q [\alpha+1]_q} \right).$$

For fixed $\alpha \in (0, \infty)$, $q \in (0, 1)$, $n \in \mathbb{N}$, and $x \in [0, \infty)$, all the quantities on the right-hand side are finite. Hence

$$|V_{n,q}^\alpha(f; x)| < \infty.$$

Therefore, $V_{n,q}^\alpha(f; x)$ is well defined for every $f \in C_\rho^*([0, \infty))$ and every $x \in [0, \infty)$. $\square$

Lemma 2.6. *Let $\alpha \in (0, \infty)$, $q \in (0, 1)$ and $n \in \mathbb{N}$, then for a given $x \in [0, \infty)$, we have*

$$V_{n,q}^\alpha(\psi_{x,1}; x) = \frac{q}{[n]_q[\alpha + 1]_q},$$

$$V_{n,q}^\alpha(\psi_{x,2}; x) = \left(\frac{x^2}{q[n]_q} \right) + \left(\frac{x}{[n]_q} \right) + \frac{q^2[2]_q}{[n]_q^2[\alpha + 2]_q[\alpha + 1]_q},$$

$$\begin{aligned} V_{n,q}^\alpha(\psi_{x,3}; x) &= \frac{q^3[3]_q[2]_q}{[n]_q^3[\alpha + 3]_q[\alpha + 2]_q[\alpha + 1]_q} + \left(1 + \frac{3q}{[\alpha + 1]_q} \right) \frac{x}{[n]_q^2} \\ &\quad + \left(\frac{3}{[\alpha + 1]_q} + \frac{1 + 2q}{q^2} + \frac{1 - q}{q} [n]_q \right) \frac{x^2}{[n]_q^2} + \left(\frac{1 + q}{q^3[n]_q^2} + \frac{1 - q}{q^2[n]_q} \right) x^3, \end{aligned}$$

$$\begin{aligned} V_{n,q}^\alpha(\psi_{x,4}; x) &= \frac{q^4[4]_q[3]_q[2]_q}{[n]_q^4[\alpha + 4]_q[\alpha + 3]_q[\alpha + 2]_q[\alpha + 1]_q} + \left(\frac{6[2]_q q^2}{[\alpha + 2]_q[\alpha + 1]_q} + \frac{4q}{[\alpha + 1]_q} + 1 \right) \frac{x}{[n]_q^3} \\ &\quad + \left(\frac{6[2]_q q}{[\alpha + 2]_q[\alpha + 1]_q} + \frac{4(1 + 2q)}{q[\alpha + 1]_q} + \frac{1 + 3q + 3q^2}{q^3} + \left(\frac{4(1 - q)}{[\alpha + 1]_q} + \frac{1 + 3q - q^2}{q^2} \right) [n]_q \right) \frac{x^2}{[n]_q^3} \\ &\quad + \left(\frac{(1 - q)^2}{q^2} [n]_q^2 + \left(\frac{1 + 4q + 3q^2 - 2q^3}{q^4} + \frac{4(1 - q)}{q[\alpha + 1]_q} \right) [n]_q + \frac{1 + 3q + 5q^2 + 3q^3}{q^5} + \frac{4(1 + q)}{q^2[\alpha + 1]_q} \right) \frac{x^3}{[n]_q^3} \\ &\quad + \left(\left(\frac{q^3 - 2q^4 + q^5}{q^6} \right) [n]_q^2 + \left(\frac{q + 3q^2 - q^4}{q^6} \right) [n]_q + \frac{1 + 2q + 2q^2 + q^3}{q^6} \right) \frac{x^4}{[n]_q^3}. \end{aligned}$$

Proof. Using Lemma 2.4, we get

$$V_{n,q}^\alpha(\psi_{x,1}; x) = \frac{q}{[n]_q[\alpha + 1]_q}.$$

For $\psi_{x,2}$, we have

$$\begin{aligned} V_{n,q}^\alpha(\psi_{x,2}; x) &= \left(\frac{[n + 1]_q}{q[n]_q} \right) x^2 + \left(\frac{2q + [\alpha + 1]_q}{[n]_q[\alpha + 1]_q} \right) x + \frac{q^2[2]_q}{[n]_q^2[\alpha + 2]_q[\alpha + 1]_q} \\ &\quad - 2x \left(x + \frac{q}{[n]_q[\alpha + 1]_q} \right) + x^2 \\ &= \left(\frac{x^2}{q[n]_q} \right) + \left(\frac{x}{[n]_q} \right) + \frac{q^2[2]_q}{[n]_q^2[\alpha + 2]_q[\alpha + 1]_q}. \end{aligned}$$

Using $[n+1]_q = 1 + q[n]_q$ and $[n+2]_q = 1 + q + q^2[n]_q$, we obtain the third central moment in the following way:

$$\begin{aligned}
V_{n,q}^\alpha(\psi_{x,3}; x) &= \frac{q^3[3]_q[2]_q}{[n]_q^3[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \left(\frac{3[2]_qq^2}{[\alpha+2]_q[\alpha+1]_q} + 1 + \frac{3q}{[\alpha+1]_q} \right) \frac{x}{[n]_q^2} \\
&\quad + \left(\frac{3[n+1]_q}{[\alpha+1]_q} + \frac{(1+2q)[n+1]_q}{q^2} \right) \frac{x^2}{[n]_q^2} + \frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} x^3 \\
&\quad - 3x \left(\left(\frac{[n+1]_q}{q[n]_q} \right) x^2 + \left(\frac{2q + [\alpha+1]_q}{[n]_q[\alpha+1]_q} \right) x + \frac{q^2[2]_q}{[n]_q^2[\alpha+2]_q[\alpha+1]_q} \right) \\
&\quad + 3x^2 \left(x + \frac{q}{[n]_q[\alpha+1]_q} \right) - x^3 \\
&= \frac{q^3[3]_q[2]_q}{[n]_q^3[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \left(1 + \frac{3q}{[\alpha+1]_q} \right) \frac{x}{[n]_q^2} \\
&\quad + \left(\frac{3[n+1]_q}{[\alpha+1]_q} + \frac{(1+2q)[n+1]_q}{q^2} - \frac{3q[n]_q}{[\alpha+1]_q} - 3[n]_q \right) \frac{x^2}{[n]_q^2} \\
&\quad + \left(\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} - \frac{3[n+1]_q}{q[n]_q} + 2 \right) x^3 \\
&= \frac{q^3[3]_q[2]_q}{[n]_q^3[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} \\
&\quad + \left(1 + \frac{3q}{[\alpha+1]_q} \right) \frac{x}{[n]_q^2} + \left(\frac{3}{[\alpha+1]_q} + \frac{1+2q}{q^2} + \frac{1-q}{q}[n]_q \right) \frac{x^2}{[n]_q^2} \\
&\quad + \left(\frac{1+q}{q^3[n]_q^2} + \frac{1-q}{q^2[n]_q} \right) x^3.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
V_{n,q}^\alpha(\psi_{x,4}; x) &= \left(\frac{4[3]_q[2]_qq^3}{[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \frac{6[2]_qq^2}{[\alpha+2]_q[\alpha+1]_q} + \frac{4q}{[\alpha+1]_q} + 1 \right) \frac{x}{[n]_q^3} \\
&\quad + \left(\frac{6[2]_qq}{[\alpha+2]_q[\alpha+1]_q} + \frac{4(1+2q)}{q[\alpha+1]_q} + \frac{(1+3q+3q^2)}{q^3} \right) \frac{[n+1]_qx^2}{[n]_q^3} \\
&\quad + \frac{q^4[4]_q[3]_q[2]_q}{[n]_q^4[\alpha+4]_q[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} \\
&\quad + \left(\frac{4}{q^2[\alpha+1]_q} + \frac{(1+2q+3q^2)}{q^5} \right) \frac{[n+1]_q[n+2]_qx^3}{[n]_q^3} \\
&\quad + \frac{[n+1]_q[n+2]_q[n+3]_q}{q^6[n]_q^3} x^4 \\
&\quad - 4x \left(\frac{q^3[3]_q[2]_q}{[n]_q^3[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \left(\frac{3[2]_qq^2}{[\alpha+2]_q[\alpha+1]_q} + 1 + \frac{3q}{[\alpha+1]_q} \right) \frac{x}{[n]_q^2} \right. \\
&\quad \left. + \left(\frac{3[n+1]_q}{[\alpha+1]_q} + \frac{(1+2q)[n+1]_q}{q^2} \right) \frac{x^2}{[n]_q^2} + \frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} x^3 \right) - 4x^3 \left(x + \frac{q}{[n]_q[\alpha+1]_q} \right) \\
&\quad + 6x^2 \left(\left(\frac{[n+1]_q}{q[n]_q} \right) x^2 + \left(\frac{2q + [\alpha+1]_q}{[n]_q[\alpha+1]_q} \right) x + \frac{q^2[2]_q}{[n]_q^2[\alpha+2]_q[\alpha+1]_q} \right) + x^4.
\end{aligned}$$

Using $[n+1]_q = 1 + q[n]_q$, $[n+2]_q = 1 + q + q^2[n]_q$ and $[n+3]_q = 1 + q + q^2 + q^3[n]_q$, we achieve

$$\begin{aligned} & V_{n,q}^\alpha(\psi_{x,4}; x) \\ &= \frac{q^4[4]_q[3]_q[2]_q}{[n]_q^4[\alpha+4]_q[\alpha+3]_q[\alpha+2]_q[\alpha+1]_q} + \left(\frac{6[2]_q q^2}{[\alpha+2]_q[\alpha+1]_q} + \frac{4q}{[\alpha+1]_q} + 1 \right) \frac{x}{[n]_q^3} \\ &+ \left(\frac{6[2]_q q}{[\alpha+2]_q[\alpha+1]_q} + \frac{4(1+2q)}{q[\alpha+1]_q} + \frac{1+3q+3q^2}{q^3} + \left(\frac{4(1-q)}{[\alpha+1]_q} + \frac{1+3q-q^2}{q^2} \right) [n]_q \right) \frac{x^2}{[n]_q^3} \\ &+ \left(\frac{(1-q)^2}{q^2} [n]_q^2 + \left(\frac{1+4q+3q^2-2q^3}{q^4} + \frac{4(1-q)}{q[\alpha+1]_q} \right) [n]_q + \frac{1+3q+5q^2+3q^3}{q^5} + \frac{4(1+q)}{q^2[\alpha+1]_q} \right) \frac{x^3}{[n]_q^3} \\ &+ \left(\left(\frac{q^3-2q^4+q^5}{q^6} \right) [n]_q^2 + \left(\frac{q+3q^2-q^4}{q^6} \right) [n]_q + \frac{1+2q+2q^2+q^3}{q^6} \right) \frac{x^4}{[n]_q^3}, \end{aligned}$$

which finishes the proof of Lemma 2.6. $\square$

We obtain Lemma 2.7 using Lemma 2.6.

Lemma 2.7. *Let $\alpha \in (0, \infty)$, $n \in \mathbb{N}$ and $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$. Then for a given $x \in [0, \infty)$, we obtain*

$$\begin{aligned} \lim_{n \rightarrow \infty} [n]_{q_n} V_{n,q_n}^\alpha(\psi_{x,1}; x) &= \frac{1}{\alpha+1}, \\ \lim_{n \rightarrow \infty} [n]_{q_n} V_{n,q_n}^\alpha(\psi_{x,2}; x) &= x^2 + x, \\ \lim_{n \rightarrow \infty} [n]_{q_n}^2 V_{n,q_n}^\alpha(\psi_{x,4}; x) &= 3x^2(1+x)^2. \end{aligned}$$

Proof. For the first central moment, we obtain

$$\lim_{n \rightarrow \infty} [n]_{q_n} V_{n,q_n}^\alpha(\psi_{x,1}; x) = \lim_{n \rightarrow \infty} [n]_{q_n} \left(\frac{q_n}{[n]_{q_n}[\alpha+1]_{q_n}} \right) = \frac{1}{\alpha+1}.$$

Similarly, we achieve

$$\lim_{n \rightarrow \infty} [n]_{q_n} V_{n,q_n}^\alpha(\psi_{x,2}; x) = \lim_{n \rightarrow \infty} [n]_{q_n} \left(\left(\frac{x^2}{q_n[n]_{q_n}} \right) + \left(\frac{x}{[n]_{q_n}} \right) + \frac{q_n^2[2]_{q_n}}{[n]_{q_n}^2[\alpha+2]_{q_n}[\alpha+1]_{q_n}} \right) = x^2 + x.$$

Since $q_n \rightarrow 1^-$ as $n \rightarrow \infty$, we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} [n]_{q_n}^2 V_{n,q_n}^\alpha(\psi_{x,4}; x) \\ &= \lim_{n \rightarrow \infty} \left(\left(\frac{1+3q_n-q_n^2}{q_n^2} \right) x^2 + \left(\frac{1+4q_n+3q_n^2-2q_n^3}{q_n^4} \right) x^3 + \left(\frac{q_n+3q_n^2-q_n^4}{q_n^6} \right) x^4 \right) \\ &+ \lim_{n \rightarrow \infty} [n]_{q_n} \left(\left(\frac{(1-q_n)^2}{q_n^2} \right) x^3 + \left(\frac{q_n^3(1-q_n)^2}{q_n^6} \right) x^4 \right). \end{aligned}$$

Using the identity $(1-q_n)[n]_{q_n} = 1 - q_n^n \leq 1$, we obtain

$$\lim_{n \rightarrow \infty} [n]_{q_n}^2 V_{n,q_n}^\alpha(\psi_{x,4}; x) = 3x^2 + 6x^3 + 3x^4 = 3x^2(1+x)^2.$$

$\square$

3 Approximation Results

To study the approximation properties of (1.3), we first recall some definitions. For $f \in C_B([0, \infty))$, the modulus of continuity is given by

$$\omega(f; \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x+h) - f(x)|.$$

For $f \in C_B([0, \infty))$, the second order modulus of smoothness is defined by

$$\omega_2(f; \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Let $C_B^2([0, \infty)) = \{p \in C_B([0, \infty)) : p', p'' \in C_B([0, \infty))\}$, then Peetre's K -functional $K(f, \delta)$ is defined as

$$K(f; \delta) = \inf_{p \in C_B^2([0, \infty))} \{\|f - p\| + \delta \|p''\|\}, \quad \delta > 0.$$

Also by [27, Theorem 2.4], there exists a constant $L > 0$ such that

$$K(f; \delta) \leq L\omega_2(f; \sqrt{\delta}). \quad (3.1)$$

Theorem 3.1. *Let $\alpha \in (0, \infty)$ and $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$. Then, for any $T_1 > 0$ and $f \in C_B([0, \infty))$, $V_{n, q_n}^\alpha(f)$ uniformly converges to f on $[0, T_1]$.*

Proof. Using Lemma 2.4 it is obvious that $V_{n, q_n}^\alpha(e_0; x) \rightarrow e_0(x)$, $V_{n, q_n}^\alpha(e_1; x) \rightarrow e_1(x)$ and $V_{n, q_n}^\alpha(e_2; x) \rightarrow e_2(x)$ uniformly. Hence, by the Korovkin theorem $V_{n, q_n}^\alpha(f)$ converges to f uniformly on $[0, T_1]$. $\square$

Lemma 3.1. *Let $\alpha \in (0, \infty)$, $f \in UC_B([0, \infty))$ and $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$, then for a given $x \in [0, \infty)$, we get*

$$|V_{n, q_n}^\alpha(f; x) - f(x)| \leq 2\omega\left(f; \sqrt{V_{n, q_n}^\alpha(\psi_{x, 2}; x)}\right).$$

Proof. Using the properties of modulus of continuity $|f(\mu) - f(x)| \leq \omega(f; |\mu - x|)$ and $\omega(f; \gamma\delta) \leq (1 + \gamma)\omega(f; \delta)$, where γ is a positive constant, we achieve

$$\begin{aligned} & |V_{n, q_n}^\alpha(f; x) - f(x)| \\ & \leq [\alpha]_{q_n} \sum_{k=0}^{\infty} v_{n, k}(q_n; x) \int_0^1 (1 - q_n \mu)_{q_n}^{(\alpha-1)} \left| f\left(\frac{q_n(q_n^{-k}[k]_{q_n} + \mu)}{[n]_{q_n}}\right) - f(x) \right| d_{q_n} \mu \\ & \leq [\alpha]_{q_n} \sum_{k=0}^{\infty} v_{n, k}(q_n; x) \int_0^1 (1 - q_n \mu)_{q_n}^{(\alpha-1)} \left(1 + \frac{1}{\delta} \left| \frac{q_n(q_n^{-k}[k]_{q_n} + \mu)}{[n]_{q_n}} - x \right| \right) \omega(f; \delta) d_{q_n} \mu \\ & \leq \left(1 + \frac{1}{\delta} [\alpha]_{q_n} \sum_{k=0}^{\infty} v_{n, k}(q_n; x) \int_0^1 (1 - q_n \mu)_{q_n}^{(\alpha-1)} \left| \frac{q_n(q_n^{-k}[k]_{q_n} + \mu)}{[n]_{q_n}} - x \right| d_{q_n} \mu \right) \omega(f; \delta). \end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$|V_{n, q_n}^\alpha(f; x) - f(x)| \leq \left(1 + \frac{1}{\delta} \sqrt{V_{n, q_n}^\alpha(\psi_{x, 2}; x)}\right) \omega(f; \delta).$$

Taking $\delta = \sqrt{V_{n, q_n}^\alpha(\psi_{x, 2}; x)}$, we obtain the desired result. $\square$

Lemma 3.2. Let $\alpha \in (0, \infty)$, $f \in C_B([0, \infty))$, $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$ and $p \in C_B^2([0, \infty))$ then for a given $x \in [0, \infty)$, we have

$$|\overline{V_{n,q_n}^\alpha}(p; x) - p(x)| \leq (V_{n,q_n}^\alpha(\psi_{x,2}; x) + (V_{n,q_n}^\alpha(\psi_{x,1}; x))^2) \|p''\|,$$

where $\overline{V_{n,q_n}^\alpha}(f; x) = V_{n,q_n}^\alpha(f; x) + f(x) - f(V_{n,q_n}^\alpha(\mu; x))$.

Proof. Let $p \in C_B^2([0, \infty))$, then by Taylor's theorem, we get

$$p(\mu) - p(x) = (\mu - x)p'(x) + \int_x^\mu (\mu - v)p''(v)dv. \quad (3.2)$$

Applying $\overline{V_{n,q_n}^\alpha}$ on (3.2) and using linearity of $\overline{V_{n,q_n}^\alpha}$, we obtain

$$\overline{V_{n,q_n}^\alpha}(p; x) - p(x) = V_{n,q_n}^\alpha \left(\int_x^\mu (\mu - v)p''(v)dv; x \right) - \int_x^{V_{n,q_n}^\alpha(\mu; x)} (V_{n,q_n}^\alpha(\mu; x) - v)p''(v)dv.$$

Using the triangle inequality and the integral inequality, we achieve

$$|\overline{V_{n,q_n}^\alpha}(p; x) - p(x)| \leq V_{n,q_n}^\alpha \left(\left| \int_x^\mu (\mu - v)p''(v)dv \right|; x \right) + \int_x^{V_{n,q_n}^\alpha(\mu; x)} |(V_{n,q_n}^\alpha(\mu; x) - v)p''(v)| dv. \quad (3.3)$$

Now,

$$\left| \int_x^\mu (\mu - v)p''(v)dv \right| \leq \frac{1}{2}(\mu - x)^2 \|p''\|.$$

As V_{n,q_n}^α is a positive linear operator, we have

$$\begin{aligned} \int_x^{V_{n,q_n}^\alpha(\mu; x)} |(V_{n,q_n}^\alpha(\mu; x) - v)p''(v)| dv &\leq \|p''\| \int_x^{V_{n,q_n}^\alpha(\mu; x)} (V_{n,q_n}^\alpha(\mu; x) - v)dv \\ &\leq (V_{n,q_n}^\alpha(\psi_{x,1}; x))^2 \|p''\|, \end{aligned} \quad (3.4)$$

and

$$V_{n,q_n}^\alpha \left(\left| \int_x^\mu (\mu - v)p''(v)dv \right|; x \right) \leq V_{n,q_n}^\alpha(\psi_{x,2}; x) \|p''\|. \quad (3.5)$$

Using equations (3.4) and (3.5) in equation (3.3), we obtain our result. $\square$

Theorem 3.2. Let $\alpha \in (0, \infty)$, $n \in \mathbb{N}$ and $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$, then for $x \in [0, \infty)$ and any $f \in C_B([0, \infty))$, there exists a real number $L_1 > 0$ such that

$$|V_{n,q_n}^\alpha(f; x) - f(x)| \leq L_1 \omega_2 \left(f; \frac{1}{2} \sqrt{(V_{n,q_n}^\alpha(\psi_{x,2}; x) + (V_{n,q_n}^\alpha(\psi_{x,1}; x))^2)} \right) + \omega(f; V_{n,q_n}^\alpha(\psi_{x,1}; x)).$$

Proof. Using triangle inequality, we have

$$\begin{aligned} |V_{n,q_n}^\alpha(f; x) - f(x)| &= |\overline{V_{n,q_n}^\alpha}(f; x) - f(x) + f(V_{n,q_n}^\alpha(\mu; x)) - f(x)| \\ &\leq |\overline{V_{n,q_n}^\alpha}(f - p; x)| + |\overline{V_{n,q_n}^\alpha}(p; x) - p(x)| + |p(x) - f(x)| \\ &\quad + |f(V_{n,q_n}^\alpha(\mu; x)) - f(x)|. \end{aligned} \quad (3.6)$$

Using the linearity of $\overline{V_{n,q_n}^\alpha}$ and the triangle inequality, we get

$$|\overline{V_{n,q_n}^\alpha}(f - p; x)| \leq |V_{n,q_n}^\alpha(f - p; x)| + |(f - p)(x)| + |(f - p)(V_{n,q_n}^\alpha(\mu; x))| \leq 3\|f - p\|. \quad (3.7)$$

Using Lemma 3.2, inequality (3.7) and the property of modulus of continuity in (3.6), we obtain

$$|V_{n,q_n}^\alpha(f; x) - f(x)| \leq 4\|f - p\| + (V_{n,q_n}^\alpha(\psi_{x,2}; x) + (V_{n,q_n}^\alpha(\psi_{x,1}; x))^2) \|p''\| + \omega(f; V_{n,q_n}^\alpha(\psi_{x,1}; x)).$$

Taking the infimum over all $p \in C_B^2([0, \infty))$ and using (3.1), we achieve

$$\begin{aligned} & |V_{n,q_n}^\alpha(f; x) - f(x)| \\ & \leq 4 \inf_{p \in C_B^2([0, \infty))} \left(\|f - p\| + \frac{(V_{n,q_n}^\alpha(\psi_{x,2}; x) + (V_{n,q_n}^\alpha(\psi_{x,1}; x))^2)}{4} \|p''\| \right) + \omega(f; V_{n,q_n}^\alpha(\psi_{x,1}; x)) \\ & = 4K \left(f; \frac{1}{4} (V_{n,q_n}^\alpha(\psi_{x,2}; x) + (V_{n,q_n}^\alpha(\psi_{x,1}; x))^2) \right) + \omega(f; V_{n,q_n}^\alpha(\psi_{x,1}; x)) \\ & \leq L_1 \omega_2 \left(f; \frac{1}{2} \sqrt{(V_{n,q_n}^\alpha(\psi_{x,2}; x) + (V_{n,q_n}^\alpha(\psi_{x,1}; x))^2)} \right) + \omega(f; V_{n,q_n}^\alpha(\psi_{x,1}; x)). \end{aligned}$$

Therefore, for any $f \in C_B([0, \infty))$, one can obtain the desired result. $\square$

In the next theorems, we get a local approximation of the class of Lipschitz functions. Lipschitz class is defined as

$$Lip_D(\xi) = \{f \in C_B([0, \infty)) : |f(x) - f(z)| \leq D|x - z|^\xi \text{ for all } x, z \in [0, \infty)\},$$

where $D > 0$ is constant and $\xi \in (0, 1]$.

Theorem 3.3. *Let $\alpha \in (0, \infty)$, $x \in [0, \infty)$, $\xi \in (0, 1]$ and $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$, then for every $f \in Lip_D(\xi)$, we achieve*

$$|V_{n,q_n}^\alpha(f; x) - f(x)| \leq D (V_{n,q_n}^\alpha(\psi_{x,2}; x))^{\frac{\xi}{2}},$$

where D is a positive constant.

Proof. Using the definition of the Lipschitz class, we obtain

$$\begin{aligned} & |V_{n,q_n}^\alpha(f; x) - f(x)| \\ & \leq V_{n,q_n}^\alpha(|f(\mu) - f(x)|; x) \\ & = [\alpha]_{q_n} \sum_{k=0}^{\infty} v_{n,k}(q_n; x) \int_0^1 (1 - q_n \mu)_{q_n}^{(\alpha-1)} \left| f\left(\frac{q_n(q_n^{-k}[k]_{q_n} + \mu)}{[n]_{q_n}}\right) - f(x) \right| d_{q_n} \mu \\ & \leq D [\alpha]_{q_n} \sum_{k=0}^{\infty} v_{n,k}(q_n; x) \int_0^1 (1 - q_n \mu)_{q_n}^{(\alpha-1)} \left| \frac{q_n(q_n^{-k}[k]_{q_n} + \mu)}{[n]_{q_n}} - x \right|^\xi d_{q_n} \mu. \end{aligned}$$

Applying Hölder's inequality and choosing $p = \frac{2}{\xi}$ and $r = \frac{2}{2-\xi}$, we have

$$|V_{n,q_n}^\alpha(f; x) - f(x)| \leq D (V_{n,q_n}^\alpha(\psi_{x,2}; x))^{\frac{\xi}{2}},$$

whence the result. $\square$

Lipschitz-type maximal function of order ξ was introduced by Lenze [28] as

$$\overline{\varphi}_\xi(f; x) = \sup_{\mu \in [0, \infty), \mu \neq x} \frac{|f(\mu) - f(x)|}{|\mu - x|^\xi}, \quad \xi \in (0, 1].$$

Theorem 3.4. Let $\alpha \in (0, \infty)$, $f \in C_B([0, \infty))$, $\xi \in (0, 1]$ and $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$, then for all $x \in [0, \infty)$, we obtain

$$|V_{n,q_n}^\alpha(f; x) - f(x)| \leq \overline{\varphi}_\xi(f; x) (V_{n,q_n}^\alpha(\psi_{x,2}; x))^{\frac{\xi}{2}}.$$

Proof. Using the definition of the Lipschitz-type maximal function, we obtain

$$|V_{n,q_n}^\alpha(f; x) - f(x)| \leq V_{n,q_n}^\alpha(|f(\mu) - f(x)|; x) \leq \overline{\varphi}_\xi(f; x) V_{n,q_n}^\alpha(|\mu - x|^\xi; x).$$

Applying Hölder's inequality and choosing $p = \frac{2}{\xi}$ and $r = \frac{2}{2-\xi}$, we achieve

$$|V_{n,q_n}^\alpha(f; x) - f(x)| \leq \overline{\varphi}_\xi(f; x) (V_{n,q_n}^\alpha(\psi_{x,2}; x))^{\frac{\xi}{2}}.$$

□

4 Weighted Approximation

Theorem 4.1. Let $\alpha \in (0, \infty)$ and $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$, then for $f \in C_\rho^*([0, \infty))$, we achieve

$$\lim_{n \rightarrow \infty} \|V_{n,q_n}^\alpha(f; \cdot) - f(\cdot)\|_\rho = 0.$$

Proof. Since $V_{n,q_n}^\alpha(e_0; x) = 1$, we have

$$\lim_{n \rightarrow \infty} \|V_{n,q_n}^\alpha(e_0; \cdot) - e_0(\cdot)\|_\rho = 0.$$

For e_1 , we obtain

$$\lim_{n \rightarrow \infty} \|V_{n,q_n}^\alpha(e_1; \cdot) - e_1(\cdot)\|_\rho \leq \lim_{n \rightarrow \infty} \left(\frac{q_n}{[n]_{q_n} [\alpha + 1]_{q_n}} \right).$$

Hence

$$\lim_{n \rightarrow \infty} \|V_{n,q_n}^\alpha(e_1; \cdot) - e_1(\cdot)\|_\rho = 0.$$

Similarly for e_2 , we achieve

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|V_{n,q_n}^\alpha(e_2; \cdot) - e_2(\cdot)\|_\rho \\ & \leq \lim_{n \rightarrow \infty} \left(\frac{1}{q_n [n]_{q_n}} + \left(\frac{2q_n + [\alpha + 1]_{q_n}}{[n]_{q_n} [\alpha + 1]_{q_n}} \right) + \frac{q_n^2 [2]_{q_n}}{[n]_{q_n}^2 [\alpha + 2]_{q_n} [\alpha + 1]_{q_n}} \right) \\ & = 0. \end{aligned}$$

By the Korovkin theorem [29] on weighted approximation, for any $f \in C_\rho^*([0, \infty))$, we get

$$\lim_{n \rightarrow \infty} \|V_{n,q_n}^\alpha(f; \cdot) - f(\cdot)\|_\rho = 0.$$

□

5 A Voronovskaya-Type Result

Theorem 5.1. *Let $\alpha \in (0, \infty)$ and $q_n \in (0, 1)$ such that $q_n \rightarrow 1^-$ as $n \rightarrow \infty$. Then for any $f \in C_\rho^*([0, \infty))$, $f', f'' \in C_\rho^*([0, \infty))$, we have*

$$\lim_{n \rightarrow \infty} [n]_{q_n} [V_{n, q_n}^\alpha(f; x) - f(x)] = \frac{1}{\alpha + 1} f'(x) + \frac{x^2 + x}{2} f''(x).$$

Proof. Using Taylor's formula, we have

$$f(\mu) - f(x) = (\mu - x)f'(x) + \frac{(\mu - x)^2}{2} f''(x) + r(\mu, x)(\mu - x)^2, \quad (5.1)$$

where $r(\cdot, x) \in C_\rho^*([0, \infty))$ is a Peano form of remainder and $\lim_{\mu \rightarrow x} r(\mu, x) = 0$. Now

$$r(\mu, x) = \frac{f(\mu) - f(x) - f'(x)(\mu - x) - \frac{1}{2}f''(x)(\mu - x)^2}{(\mu - x)^2}, \quad \mu \neq x.$$

Since $f \in C_\rho^*([0, \infty)) \subset C_\rho([0, \infty))$, there exists $M_f > 0$ such that $|f(\mu)| \leq M_f(1 + \mu^2)$. For fixed x and $\mu \geq \max\{1, 2x\}$, we have $(\mu - x)^2 \geq \mu^2/4$ and

$$M_f(1 + \mu^2) + |f(x)| + |f'(x)|(\mu + x) + \frac{1}{2}|f''(x)|(\mu - x)^2 \leq A'_x(1 + \mu + \mu^2),$$

for some constant $A'_x > 0$ independent of μ . Hence

$$|r(\mu, x)| \leq 4A'_x \left(1 + \frac{1}{\mu} + \frac{1}{\mu^2}\right).$$

Thus, $|r(\mu, x)| \leq 4C'_x$ with $C'_x = 3A'_x$ for all sufficiently large μ . By the continuity of $r(\cdot, x)$, it follows that $|r(\mu, x)| \leq C_x$ for all $\mu \geq 0$, for some $C_x > 0$. Hence

$$0 \leq \lim_{\mu \rightarrow \infty} \frac{|r(\mu, x)|^2}{1 + \mu^2} \leq \lim_{\mu \rightarrow \infty} \frac{C_x^2}{1 + \mu^2} = 0.$$

Thus,

$$r^2(\cdot, x) \in C_\rho^*([0, \infty)).$$

Applying the operator V_{n, q_n}^α to both sides of the equation (5.1), we obtain

$$V_{n, q_n}^\alpha(f; x) - f(x) = f'(x)V_{n, q_n}^\alpha(\psi_{x,1}; x) + \frac{f''(x)}{2}V_{n, q_n}^\alpha(\psi_{x,2}; x) + V_{n, q_n}^\alpha(r(\mu, x)(\mu - x)^2; x). \quad (5.2)$$

Using the Cauchy-Schwarz's inequality, we have

$$|V_{n, q_n}^\alpha(r(\mu, x)(\mu - x)^2; x)| \leq \sqrt{V_{n, q_n}^\alpha(\psi_{x,4}; x)} \sqrt{V_{n, q_n}^\alpha(r^2(\mu, x); x)}.$$

Multiplying both sides by $[n]_{q_n}$, we get

$$[n]_{q_n} |V_{n, q_n}^\alpha(r(\mu, x)(\mu - x)^2; x)| \leq [n]_{q_n} \sqrt{V_{n, q_n}^\alpha(\psi_{x,4}; x)} \sqrt{V_{n, q_n}^\alpha(r^2(\mu, x); x)}.$$

Using Lemma 2.7, as $q_n \rightarrow 1^-$, we have

$$\lim_{n \rightarrow \infty} [n]_{q_n} \sqrt{V_{n, q_n}^\alpha(\psi_{x,4}; x)} = \sqrt{3}x(1 + x).$$

Moreover, since

$$\lim_{n \rightarrow \infty} V_{n,q_n}^\alpha(r^2(\mu, x); x) = r^2(x, x) = 0,$$

we obtain

$$\lim_{n \rightarrow \infty} [n]_{q_n} |V_{n,q_n}^\alpha(r(\mu, x)(\mu - x)^2; x)| = 0.$$

Taking limits in equation (5.2) and using Lemma 2.7, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} [n]_{q_n} [V_{n,q_n}^\alpha(f; x) - f(x)] \\ &= \lim_{n \rightarrow \infty} \left(\frac{q_n}{[\alpha + 1]_{q_n}} f'(x) + \left(\left(\frac{x^2}{q_n} \right) + x + \frac{q_n^2 [2]_{q_n}}{[n]_{q_n} [\alpha + 2]_{q_n} [\alpha + 1]_{q_n}} \right) \frac{f''(x)}{2} \right) \\ &= \frac{1}{\alpha + 1} f'(x) + \frac{x^2 + x}{2} f''(x). \end{aligned}$$

This completes the proof. $\square$

6 Graphical Representation and Numerical Comparison

Example 6.1. Consider the function $f(x) = 7x^2 - x$, $x \in [0, 5]$. Here we take $q = q_n = 1 - \frac{1}{n^2}$, and vary the values of α and n .

Table 2: Maximum error $\|V_{n,q_n}^\alpha(f; \cdot) - f(\cdot)\|_\infty$ on $[0, 5]$ for $\alpha \in \{1, 10, 100\}$ and $n \in \{100, 500, 1000\}$

	$n = 100$	$n = 500$	$n = 1000$
$\alpha = 1$	2.4575	0.4895	0.2446
$\alpha = 10$	2.1737	0.4330	0.2164
$\alpha = 100$	2.1175	0.4218	0.2108

From Table 2, we can observe that the numerical results support the theoretical findings. The convergence of operators $V_{n,q_n}^{10}(f; x)$ to $f(x)$ for different values of n is illustrated in Figure 1.

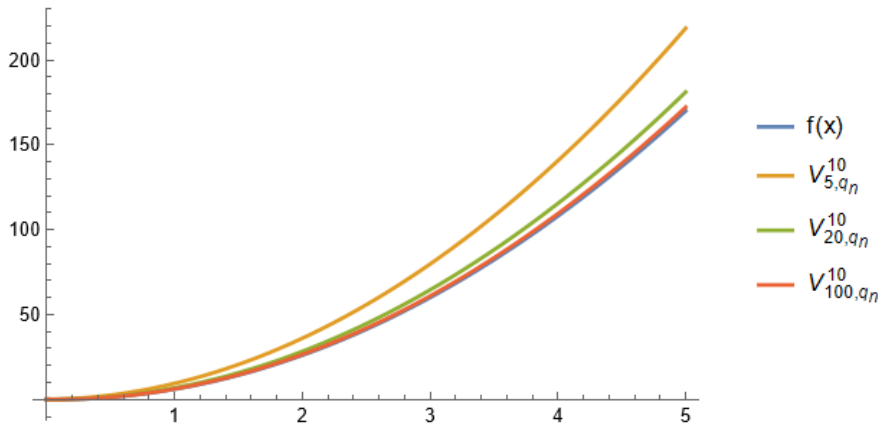


Figure 1: Approximation to $f(x)$ by $V_{n,q_n}^{10}(f; x)$ for $n \in \{5, 20, 100\}$

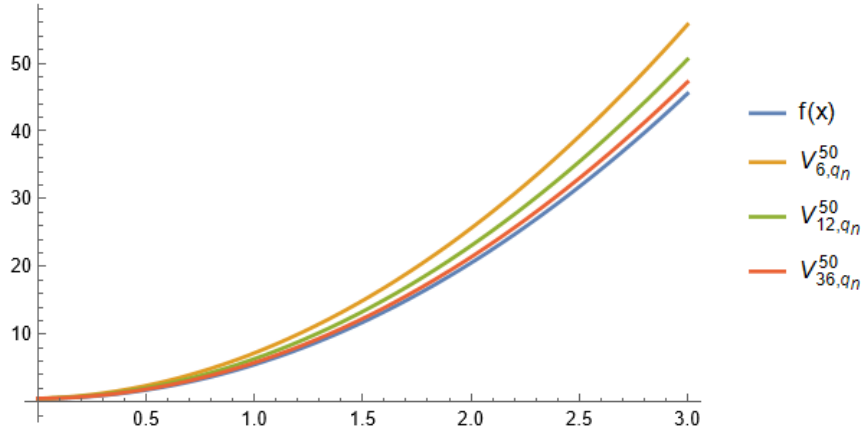


Figure 2: Approximation to $f(x)$ by $V_{n,q_n}^{50}(f; x)$ for $n \in \{6, 12, 36\}$

Example 6.2. Consider the function $f(x) = 5x^2 + \frac{1}{2}$, $x \in [0, 3]$. Here we take $q = q_n = 1 - \frac{1}{e^n}$, and changed the value of α and n . The convergence of operators $V_{n,q_n}^{50}(f; x)$ to $f(x)$ for different values of n is illustrated in Figure 2. Figure 2 shows that the approximation to the function via operators q -RLBaK improves as the value of n increases. In other words, the error is minimum for the largest value of n .

Table 3: Maximum error $\|V_{n,q_n}^\alpha(f; \cdot) - f(\cdot)\|_\infty$ on $[0, 3]$ for $\alpha \in \{0.5, 5, 50\}$ and $n \in \{6, 12, 36, 100, 500, 1000\}$

	$n = 6$	$n = 12$	$n = 36$	$n = 100$	$n = 500$	$n = 1000$
$\alpha = 0.5$	13.5034	6.6854	2.2243	0.8003	0.1600	0.0800
$\alpha = 5$	10.9293	5.4185	1.8057	0.6500	0.1300	0.0650
$\alpha = 50$	10.1857	5.0492	1.6830	0.6059	0.1212	0.0606

The results in Table 3 provide numerical evidence supporting the theoretical results.

7 Conclusion

This paper presents the fractional version of Baskakov–Kantorovich operators as novel generalizations in the context of q -calculus. We established their essential approximation characteristics by calculating moments and central moments, and verified Korovkin-type convergence results. Using the Lipschitz class, Lipschitz-type maximal functions, Peetre’s K -functional, and modulus of smoothness techniques, we obtained quantitative estimates of the rate of convergence. Further understanding of the operators’ local approximation behavior was made possible by the Voronovskaya-type asymptotic theorem and the weighted approximation results showed that they work well on unbounded intervals. The numerical and graphical results illustrate convergence for the selected test functions as n increases. The tabulated errors also show that larger values of α yield smaller maximum approximation errors for the examples and parameter values considered.

8 Future Work

The examination of q -fractional Kantorovich-type operators opens several promising directions for future research. One potential extension involves the creation of Stancu variants of the proposed operators to introduce additional shape parameters. Another interesting path is to create bivariate or multivariate analogues for use in surface approximation. It may also be useful to look into statistical and weighted statistical convergence properties to make them more useful in probabilistic approximation theory. Additionally, examining these operators within the framework of (p, q) -calculus may provide more profound structural insights. Applications to numerical solutions of fractional differential and integral equations constitute a significant area for future investigation.

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