Approx. Theory Spec. Funct. 2 (1) (2026) 55–71
https://doi.org/10.65135/atsf.2026.10
https://atsf.com.tr

APPROXIMATION THEORY and SPECIAL FUNCTIONS
[ISSN: 3108-5083 (Online)]

Research Article

Oscillatory Behavior and Qualitative Properties of Second-Order Nonlinear Neutral Differential Equations of Noncanonical Type

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Abstract: This study examines a class of neutral differential equations involving both delayed and advanced arguments, in which the present state of the system depends simultaneously on its past and future values. Such equations arise naturally in the mathematical modeling of various physical, biological, and engineering phenomena. By employing a refined analytical framework, new sufficient conditions are derived to ensure that all solutions exhibit oscillatory behavior. The proposed criteria generalize and improve several existing results by relaxing restrictive assumptions commonly imposed in the literature, thereby providing a more comprehensive approach to the qualitative analysis of these systems. A set of illustrative examples is presented to demonstrate the sharpness and effectiveness of the obtained conditions, highlighting their relevance for a deeper understanding of oscillatory dynamics in nonlinear neutral differential equations.

Keywords: Oscillation, neutral differential equations, noncanonical case, asymptotic behavior.

AMS Math. Subject Class. (2020): 34C10, 34K11.

1 Introduction

Mathematics has long been considered the language of nature and an essential tool for the mind to explore dynamic cosmic phenomena. Among its most influential branches, differential equations (DEs) are the cornerstone of analyzing dynamical systems and formulating scientific laws. They enable us to express relationships between changes and understand the qualitative patterns that govern the evolution of behavior over time. These equations have formed the mathematical foundation for understanding complex and nonlinear behavior in physics, engineering, biology, and economics, giving them a pivotal position in both mathematical theory and practical applications [1, 2].

Within this broad framework, neutral differential equations (NDEs) occupy a special position, distinguished by their ability to connect the current values of a function and its derivatives, on the one hand, and delay or advanced values, on the other. This property makes them an ideal tool for studying systems with temporal memory or extended self-dependence, enabling the description of

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Received: 25.01.2026; *Accepted*: 09.03.2026; *Published*: 08.04.2026

phenomena that cannot be explained by conventional differential equations, such as the vibration of elastic mechanical systems, the behavior of electrical transmission circuits, and some biological processes that exhibit mutual time dependence [3–5].

Oscillations and their associated qualitative behaviors are a fundamental focus of the study of these models, as they reflect recurring or periodic dynamics that express equilibria and transitions in natural and engineered systems. Recent research efforts have contributed to the development of a precise set of new conditions and criteria for testing oscillatory and qualitative behavior in neutral equations, enabling the construction of more in-depth theories that deal with the non-standard and complex forms of these equations [6–9].

In this study, we examine the oscillatory behavior of solutions to the second-order nonlinear neutral differential equation

$$\left(\kappa(\nu)\omega'(\nu)\right)' + \int_a^b \mathfrak{q}(\nu,\ell)\,\varkappa^{\alpha}(\varrho(\nu,\ell))\,\mathrm{d}\ell = 0,\ \ \nu \geq \nu_0, \tag{1.1}$$

where $\omega(\nu) = \varkappa(\nu) + \mathfrak{p}(\nu)\varkappa(\eta(\nu))$. The following hypotheses are assumed throughout this study:

(H$_1$) $0 < \alpha \leq 1$, α is a ratio of odd positive integers;

(H$_2$) $\mathfrak{q} \in C([\nu_0,\infty) \times (\kappa, b), \mathbb{R})$, and $\mathfrak{q}(\nu,\ell) \geq 0$;

(H$_3$) $\mathfrak{p} \in C([\nu_0,\infty),\mathbb{R}^+)$, $0 \leq \mathfrak{p}(\nu) \leq d < 1$, $\eta \in C^1([\nu_0,\infty),\mathbb{R})$, $\varrho \in C^1([\nu_0,\infty) \times (\kappa,b),\mathbb{R})$, $\eta(\nu) \leq \nu$, ϱ has nonnegative partial derivative with respect to ν and nondecreasing with respect to ℓ, $\lim_{\nu\to\infty}\eta(\nu) = \infty$, $\lim_{\nu\to\infty}\varrho(\nu,\ell) = \infty$ for $\ell \in [a,b]$;

(H$_4$) $\kappa(\nu) \in C([\nu_0,\infty),\mathbb{R}^+)$, satisfying the condition:

$$\mathcal{R}(\nu) = \int_\nu^\infty \frac{1}{\kappa(\varsigma)}\mathrm{d}\varsigma,\ \ \nu \geq \nu_0,$$

with

$$\mathcal{R}(\nu_0) < \infty. \tag{1.2}$$

indicating that the equation is in noncanonical form. By a solution of Eq. (1.1), we mean a function $\varkappa(\nu) \in C([\nu_\varkappa,\infty),\mathbb{R})$ for some $\nu_\varkappa \geq \nu_0$ such that $\omega(\nu) \in C^1([\nu_\varkappa,\infty),\mathbb{R})$, $\kappa(\nu)\omega'(\nu) \in C^1([\nu_\varkappa,\infty),\mathbb{R})$, and $\varkappa(\nu)$ satisfies (1.1) on $[\nu_\varkappa,\infty)$. We take into account just those solutions of (1.1) that exist on some half-line $[\nu_\varkappa,\infty)$ and satisfy

$$\sup\{|\varkappa(\nu)| : \nu \geq V\} > 0 \quad \text{for any} \quad V \geq \nu_\varkappa;$$

we also assume implicitly that (1.1) has such solutions. An oscillatory solution $\varkappa(\nu)$ of (1.1) is one that has arbitrarily large zeros on $[\nu_\varkappa,\infty)$, meaning that for any $\nu_1 \in [\nu_\varkappa,\infty)$, there exists $\nu_2 \geq \nu_1$ such that $\varkappa(\nu_2) = 0$; if not, it is referred to as nonoscillatory, that is, if it is eventually positive or eventually negative. If every solution to Eq. (1.1) oscillates, then the equation is considered oscillatory.

The work of Han et al. [10], Baculikova [11], Bazighifan et al. [12], Moaaz et al. [13], Muhib [14] and Aldiaiji et al. [15, 16] demonstrates the growing interest in investigating the oscillatory and exponential behavior of differential equations with delays and neutral boundaries across different orders. This research momentum has enabled significant progress in characterizing complex periodic solutions, ranging from simple harmonic oscillations to chaotic patterns, and has enabled accurate evaluations of fundamental properties such as amplitude, frequency, and stability. The following is a review of the most prominent studies that have contributed to this trend:

Grammatikopoulos et al. [17] showed that the assumption $0 \leq \mathfrak{p}(\nu) \leq 1$ with

$$\int_{\nu_0}^{\infty} \mathfrak{q}(\varsigma)(1-\mathfrak{p}(\varsigma-\varrho))\, d\varsigma = \infty$$

is sufficient to guarantee the oscillation of the neutral equation

$$(\varkappa(\nu) + \mathfrak{p}(\nu)\varkappa(\nu-\eta))'' + \mathfrak{q}(\nu)\varkappa(\nu-\varrho) = 0.$$

For the same equation, Erbe et al. [18] presented an oscillation criterion based on $\mathfrak{q}(\nu) \geq \mathfrak{q}_0 > 0$, $\mathfrak{p}_1 \leq \mathfrak{p}(\nu) \leq \mathfrak{p}_2$, with the condition $\mathfrak{p}(\nu)$ not eventually negative. Later researchers improved these results and generalized them to broader cases. For example, Grace and Lalli [19] studied the oscillation in the equation

$$\left(\kappa(\nu)(\varkappa(\nu) + \mathfrak{p}(\nu)\varkappa(\nu-\eta))'\right)' + \mathfrak{q}(\nu) f(\varkappa(\nu-\varrho)) = 0,$$

with the condition $f(\varkappa)/\varkappa \geq k > 0$ and $\int_{\nu_0}^{\infty} 1/\kappa(\varsigma) d\varsigma = \infty$. Similarly, Baculikova and Dzurina [20] investigated a neutral equation of the form

$$\left(\kappa(\nu)(\varkappa(\nu) + \mathfrak{p}(\nu)\varkappa(\eta(\nu)))'\right)' + \mathfrak{q}(\nu)\varkappa(\varrho(\nu)) = 0,$$

using the assumption $0 \leq \mathfrak{p}(\nu) \leq \mathfrak{p}_0 < \infty$ and $\eta \circ \varrho = \varrho \circ \eta$, their results were based on recent comparative theorems that facilitate the reduction of second-order oscillations to first-order equations. In a later step, Grace et al. [21] investigated the oscillation in the nonlinear equation

$$\left(\kappa(\nu)(\varkappa(\nu) + \mathfrak{p}(\nu)\varkappa(\eta(\nu)))'\right)' + \mathfrak{q}(\nu)\varkappa^{\alpha}(\varrho(\nu)) = 0,$$

and proved that all solutions in this case are oscillatory. On the other hand, Agarwal et al. [22], Bohner et al. [23, 24], Tunç et al. [25] and Jadlovská et al. [26] presented new results on quasilinear equations of the form

$$\left(\kappa(\nu)\left[(\varkappa(\nu) + \mathfrak{p}(\nu)\varkappa(\eta(\nu)))'\right]^{\alpha}\right)' + \mathfrak{q}(\nu)\varkappa^{\alpha}(\varrho(\nu)) = 0$$

by establishing novel oscillation conditions when $\int_{\nu_0}^{\infty} 1/\kappa^{\frac{1}{\alpha}}(\varsigma) d\varsigma < \infty$.
Grace and Graef [27] also studied the oscillation of the equation

$$\left(\kappa(\nu)\left[\omega'(\nu)\right]^{\alpha}\right)' + \mathfrak{q}(\nu)\varkappa^{\beta}(\eta(\nu)) = 0,$$

developing new criteria to reduce nonlinear neutral solutions to simpler, more analyzable forms.
Li et al.[28] examined sufficient conditions for oscillation in the equation

$$\left(\kappa(\nu)(\omega'(\nu))^{\alpha}\right)' + \mathfrak{q}(\nu)\varkappa^{\beta}(\varrho(\nu)) = 0,$$

expanding the scope of the study to include both delay and advanced arguments.
In a broader context, Alrashdi et al. [29] presented a study on the oscillation properties of equations of the form

$$\left(\kappa(\nu)\left[\omega'(\nu)\right]^{\alpha}\right)' + \mathfrak{q}(\nu)\varkappa^{\beta}(\varrho(\nu)) = 0,$$

proposing sufficient conditions for the oscillation of all solutions and adding new monotonic properties for iterative solutions. Finally, Masood et al. [30] focused on equations of the type:

$$\left(\kappa(\nu)(\omega'(\nu))^{\alpha}\right)' + \int_a^b \mathfrak{q}(\nu,\ell)\varkappa^{\beta}(\varrho(\nu,\ell))\, d\ell = 0.$$

They relied on iterative techniques to derive oscillation criteria, extending previous results and adding new insights into the stability and asymptotic behavior of solutions.

These accumulated efforts clearly highlight the continuous development of oscillation theories in neutral differential equations and illustrate how the development of analytical tools and iterative techniques has opened new horizons for studying this class of equations. However, many of the available results are still limited to specific cases or subject to restrictive assumptions about the equation's coefficients or the nature of the temporal arguments. Hence, this is a research gap that the present work seeks to fill. Based on this background, this research aims to study Equation (1.1) as a general model for second-order neutral differential equations by developing new oscillation criteria based on modern comparison techniques, as adopted in previous studies [21]. These results are expected to contribute to a deeper understanding of the asymptotic behavior of solutions and to expand the scope of application of oscillation theories to accommodate the increasing complexity of contemporary dynamical systems.

In the sequel, we assume that all functional inequalities hold for all ν large enough. We only deal with positive solutions of (1.1) without loss of generality, since if $\varkappa(\nu)$ is a solution of (1.1), then $-\varkappa(\nu)$ is a solution as well.

The subsequent sections are organized as follows: Section 2 introduces several theorems providing criteria that ensure oscillation for all solutions of Eq. (1.1). In Section 3, illustrative examples are presented to clarify the theoretical results. Finally, Section 4 concludes the paper with a summary of the main findings, concluding remarks, and suggestions for future research.

2 Main Results

For clarity and convenience, we introduce the following notation:

$$\omega(\nu) = \varkappa(\nu) + \mathfrak{p}(\nu)\,\varkappa(\eta(\nu))$$

and

$$\widetilde{\mathfrak{q}}(\nu) = \int_a^b \mathfrak{q}(\nu,\ell)\,\mathrm{d}\ell.$$

Additionally, we define

$$\widehat{\mathfrak{q}}(\nu) = \int_a^b \mathfrak{q}(\nu,\ell)\left(1 - \mathfrak{p}(\varrho(\nu,\ell))\,\frac{\mathcal{R}(\eta(\varrho(\nu,\ell)))}{\mathcal{R}(\varrho(\nu,\ell))}\right)^{\alpha}\mathrm{d}\ell,\ \nu \geq \nu_1 \geq \nu_0.$$

It is important to note that $\widehat{\mathfrak{q}}(\nu) \geq 0$ for all $\nu \geq \nu_1 \geq \nu_0$.

Remark 2.1. *For Eq. (1.1), we begin by presenting three oscillation conditions corresponding to the case where $\varrho(\nu,\ell)$ represents a delay argument, i.e., $\varrho(\nu,\ell) \leq \nu$.*

Theorem 2.1. *Let (1.2) hold. If*

$$\int_{\nu_0}^{\infty} \mathcal{R}(\varsigma)\,\widetilde{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma = \infty \tag{2.1}$$

and

$$\limsup_{\nu\to\infty}\left(\mathcal{R}(\nu)\int_{\nu_0}^{\nu}\widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma + \mathcal{R}^{-\alpha}(\varrho(\nu,b))\int_{\nu}^{\infty}\mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma\right) > \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases} \tag{2.2}$$

then Eq. (1.1) is oscillatory.

Proof. Let $\varkappa(\nu)$ be a nonoscillatory solution of Eq. (1.1), say $\varkappa(\nu) > 0$, $\varkappa(\eta(\nu)) > 0$, and $\varkappa(\varrho(\nu,\ell)) > 0$ for $\nu \geq \nu_1$ for some $\nu_1 \geq \nu_0$. From (1.1), we may then observe that

$$\left(\kappa(\nu)\omega'(\nu)\right)' = -\int_a^b \mathfrak{q}(\nu,\ell)\,\varkappa^{\alpha}(\varrho(\nu,\ell))\,\mathrm{d}\ell \leq 0 \quad \text{for} \quad \nu \geq \nu_1, \tag{2.3}$$

which implies that $\kappa(\nu)\omega'(\nu)$ is nonincreasing and eventually of one sign; thus, there exists $\nu_2 \geq \nu_1$ such that, for $\nu \geq \nu_2$, either

$$\begin{aligned} (\mathrm{N}_1)\ \omega(\nu) &> 0,\ \omega'(\nu) > 0,\ \text{and}\ \left(\kappa(\nu)\omega'(\nu)\right)' \leq 0,\ \text{or} \\ (\mathrm{N}_2)\ \omega(\nu) &> 0,\ \omega'(\nu) < 0,\ \text{and}\ \left(\kappa(\nu)\omega'(\nu)\right)' \leq 0. \end{aligned}$$

First, we look at case (N_1). We may observe from the definition of $\omega(\nu)$, that

$$\varkappa(\nu) = \omega(\nu) - \mathfrak{p}(\nu)\,\varkappa(\eta(\nu)) \geq \omega(\nu) - d\cdot\omega(\eta(\nu)) \geq \omega(\nu) - d\cdot\omega(\nu) \geq (1-d)\,\omega(\nu). \tag{2.4}$$

Using (2.4) in (1.1) yields

$$\begin{aligned} \left(\kappa(\nu)\omega'(\nu)\right)' &= -\int_a^b \mathfrak{q}(\nu,\ell)\,\varkappa^{\alpha}(\varrho(\nu,\ell))\,\mathrm{d}\ell \\ &\leq -(1-d)^{\alpha}\int_a^b \mathfrak{q}(\nu,\ell)\,\omega^{\alpha}(\varrho(\nu,\ell))\,\mathrm{d}\ell. \end{aligned}$$

Since $\omega(\nu) \in (\mathrm{N}_1)$ and $a \leq \ell \leq b$, then $\omega(\nu)$ is increasing and $\omega(\varrho(\nu,a)) \leq \omega(\varrho(\nu,\ell)) \leq \omega(\varrho(\nu,b))$. Then

$$\begin{aligned} \left(\kappa(\nu)\omega'(\nu)\right)' &\leq -(1-d)^{\alpha}\,\omega^{\alpha}(\varrho(\nu,a))\int_a^b \mathfrak{q}(\nu,\ell)\,\mathrm{d}\ell, \\ &= -(1-d)^{\alpha}\,\widetilde{\mathfrak{q}}(\nu)\,\omega^{\alpha}(\varrho(\nu,a)), \end{aligned}$$

or equivalently

$$\left(\kappa(\nu)\omega'(\nu)\right)' + (1-d)^{\alpha}\,\widetilde{\mathfrak{q}}(\nu)\,\omega^{\alpha}(\varrho(\nu,a)) \leq 0, \tag{2.5}$$

for $\nu \geq \nu_3$ for some $\nu_3 \geq \nu_2$. Since $\omega(\nu) > 0$ and $\omega'(\nu) > 0$ there exists a constant $c > 0$ such that $\omega(\nu) \geq c > 0$ for all $\nu \geq \nu_3$. Consequently, inequality (2.5) can be rewritten as

$$\left(\kappa(\nu)\omega'(\nu)\right)' + c^{\alpha}(1-d)^{\alpha}\,\widetilde{\mathfrak{q}}(\nu) \leq 0 \quad \text{for} \quad \nu \geq \nu_3. \tag{2.6}$$

Integrating the inequality (2.6) from ν_3 to ν, we get

$$\omega'(\nu) \leq \frac{\kappa(\nu_3)\omega'(\nu_3)}{\kappa(\nu)} - c^{\alpha}(1-d)^{\alpha}\,\frac{1}{\kappa(\nu)}\int_{\nu_3}^{\nu}\widetilde{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma.$$

Also, by integrating the last inequality from ν_3 to ν, we get

$$\omega(\nu) \leq \omega(\nu_3) + \int_{\nu_3}^{\nu}\frac{\kappa(\nu_3)\omega'(\nu_3)}{\kappa(\varsigma)}\mathrm{d}\varsigma - c^{\alpha}(1-d)^{\alpha}\int_{\nu_3}^{\nu}\frac{1}{\kappa(u)}\int_{\nu_3}^{u}\widetilde{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma\mathrm{d}u \to -\infty,\ \text{as}\ \nu\to\infty,$$

which contradicts the positivity of $\omega(\nu)$.
Next, assume that case (N_2) holds. In this case, there exists a constant $k \geq 0$ such that

$$\lim_{\nu\to\infty}\omega(\nu) = k < \infty.$$

We propose that $k=0$. To derive a contradiction, let us assume $k>0$. There exist $\nu_3 \geq \nu_2$ and $\lambda>1$ satisfying $\lambda d<1$ such that

$$k<\omega\left(\nu\right) \leq k\lambda, \tag{2.7}$$

for $\nu \geq \nu_3$. Now,

$$\varkappa\left(\nu\right)=\omega\left(\nu\right)-\mathfrak{p}\left(\nu\right)\varkappa\left(\eta\left(\nu\right)\right) \geq \omega\left(\nu\right)-d\cdot\omega\left(\eta\left(\nu\right)\right) \geq \left(\frac{1-\lambda d}{\lambda}\right)\omega\left(\nu\right). \tag{2.8}$$

Using (2.8) in (1.1) yields

$$\begin{aligned} \left(\kappa(\nu)\omega'(\nu)\right)' &= -\int_a^b \mathfrak{q}\left(\nu,\ell\right)\varkappa^{\alpha}\left(\varrho\left(\nu,\ell\right)\right)\mathrm{d}\ell \\ &\leq -\left(\frac{1-\lambda d}{\lambda}\right)^{\alpha}\int_a^b \mathfrak{q}\left(\nu,\ell\right)\omega^{\alpha}\left(\varrho\left(\nu,\ell\right)\right)\mathrm{d}\ell. \end{aligned}$$

Since $\omega\left(\nu\right) \in (\mathrm{N}_2)$ and $a \leq \ell \leq b$, then $\omega\left(\nu\right)$ is decreasing and $\omega\left(\varrho\left(\nu,a\right)\right) \geq \omega\left(\varrho\left(\nu,\ell\right)\right) \geq \omega\left(\varrho\left(\nu,b\right)\right)$. Then

$$\begin{aligned} \left(\kappa(\nu)\omega'(\nu)\right)' &\leq -\left(\frac{1-\lambda d}{\lambda}\right)^{\alpha}\omega^{\alpha}\left(\varrho\left(\nu,b\right)\right)\int_a^b \mathfrak{q}\left(\nu,\ell\right)\mathrm{d}\ell \\ &= -\left(\frac{1-\lambda d}{\lambda}\right)^{\alpha}\widetilde{\mathfrak{q}}\left(\nu\right)\omega^{\alpha}\left(\varrho\left(\nu,b\right)\right), \end{aligned}$$

or equivalently

$$\left(\kappa(\nu)\omega'(\nu)\right)'+\left(\frac{1-\lambda d}{\lambda}\right)^{\alpha}\widetilde{\mathfrak{q}}\left(\nu\right)\omega^{\alpha}\left(\varrho\left(\nu,b\right)\right) \leq 0, \tag{2.9}$$

for some $\nu_4 \geq \nu_3$, for $\nu \geq \nu_4$. Integrating (2.9) from ν_4 to ν leads to

$$-\kappa(\nu)\omega'(\nu) \geq k^{\alpha}\left(\frac{1-\lambda d}{\lambda}\right)^{\alpha}\int_{\nu_4}^{\nu}\widetilde{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma. \tag{2.10}$$

Integrating again and using (2.1) yields

$$\begin{aligned} \omega\left(\nu_4\right) &\geq k^{\alpha}\left(\frac{1-\lambda d}{\lambda}\right)^{\alpha}\int_{\nu_4}^{\infty}\frac{1}{\kappa(u)}\int_{\nu_4}^{u}\widetilde{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma\mathrm{d}u \\ &= k^{\alpha}\left(\frac{1-\lambda d}{\lambda}\right)^{\alpha}\int_{\nu_4}^{\infty}\mathcal{R}\left(\varsigma\right)\widetilde{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma=\infty, \end{aligned}$$

which leads to a contradiction. Hence, we must have

$$\lim_{\nu\to\infty}\omega\left(\nu\right)=0.$$

From case (N_2), it may be concluded that

$$\omega\left(\nu\right) \geq -\int_{\nu}^{\infty}\frac{\kappa(\varsigma)\omega'(\varsigma)}{\kappa(\varsigma)}\mathrm{d}\varsigma \geq -\kappa(\nu)\omega'(\nu)\int_{\nu}^{\infty}\frac{1}{\kappa(\varsigma)}\mathrm{d}\varsigma=-\mathcal{R}\left(\nu\right)\kappa(\nu)\omega'(\nu),$$

and so

$$\mathcal{R}\left(\nu\right)\kappa(\nu)\omega'(\nu)+\omega\left(\nu\right) \geq 0, \tag{2.11}$$

which implies that

$$\left(\frac{\omega(\nu)}{\mathcal{R}(\nu)}\right)' = \frac{\mathcal{R}(\nu)\,\kappa(\nu)\omega'(\nu)+\omega(\nu)}{\kappa(\nu)\mathcal{R}^2(\nu)} \geq 0,$$

i.e., $\omega(\nu)/\mathcal{R}(\nu)$ is eventually nondecreasing, say for $\nu \geq \nu_5$ for some $\nu_5 \geq \nu_4$. We can see from this and the definition of $\omega(\nu)$, that

$$\varkappa(\nu) \geq \omega(\nu) - \mathfrak{p}(\nu)\,\omega(\eta(\nu)) \geq \left(1-\mathfrak{p}(\nu)\frac{\mathcal{R}(\eta(\nu))}{\mathcal{R}(\nu)}\right)\omega(\nu). \tag{2.12}$$

Using (2.12) in (1.1) gives

$$\left(\kappa(\nu)\omega'(\nu)\right)' + \omega^{\alpha}(\varrho(\nu,b))\int_a^b \mathfrak{q}(\nu,\ell)\left[1-\mathfrak{p}(\varrho(\nu,\ell))\frac{\mathcal{R}[\eta(\varrho(\nu,\ell))]}{\mathcal{R}(\varrho(\nu,\ell))}\right]^{\alpha}\mathrm{d}\ell \leq 0,$$

or equivalent

$$\left(\kappa(\nu)\omega'(\nu)\right)' + \widehat{\mathfrak{q}}(\nu)\,\omega^{\alpha}(\varrho(\nu,b)) \leq 0, \tag{2.13}$$

it is equivalently expressed in writing

$$\left[\kappa(\nu)\omega'(\nu)\mathcal{R}(\nu)+\omega(\nu)\right]' + \mathcal{R}(\nu)\,\widehat{\mathfrak{q}}(\nu)\,\omega^{\alpha}(\varrho(\nu,b)) \leq 0. \tag{2.14}$$

Integrating (2.14) from ν to u, letting $u \to \infty$, and using (2.11) yields

$$\kappa(\nu)\omega'(\nu)\mathcal{R}(\nu)+\omega(\nu) \geq \int_{\nu}^{\infty}\mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma. \tag{2.15}$$

When (2.13) is integrated from ν_5 to ν and multiplication by $\mathcal{R}(\nu)$ gives

$$-\kappa(\nu)\omega'(\nu)\mathcal{R}(\nu) \geq \mathcal{R}(\nu)\int_{\nu_5}^{\nu}\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma. \tag{2.16}$$

From (2.15) and (2.16), it follows that

$$\omega(\nu) \geq \mathcal{R}(\nu)\int_{\nu_5}^{\nu}\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma + \int_{\nu}^{\infty}\mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma. \tag{2.17}$$

Given that $\omega(\nu)$ is decreasing, we may observe that for $\varsigma \leq \nu$, $\varrho(\varsigma,b) \leq \varrho(\nu,b) \leq \nu$, and so $\omega(\varrho(\varsigma,b)) \geq \omega(\varrho(\nu,b)) \geq \omega(\nu)$ for $\nu \geq \nu_5$. Consequently,

$$\int_{\nu_5}^{\nu}\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \geq \omega^{\alpha}(\nu)\int_{\nu_5}^{\nu}\widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma. \tag{2.18}$$

Also, for $\varsigma > \nu$, we have $\varrho(\varsigma,b) \geq \varrho(\nu,b)$ and $\varrho(\nu,b) \leq \nu$. With $\omega(\nu)$ decreasing and $\omega(\nu)/\mathcal{R}(\nu)$ nondecreasing, we find

$$\frac{\omega^{\alpha}(\varrho(\varsigma,b))}{\mathcal{R}^{\alpha}(\varrho(\varsigma,b))} \geq \frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))} \geq \frac{\omega^{\alpha}(\nu)}{\mathcal{R}^{\alpha}(\varrho(\nu,b))}.$$

Thus,

$$\begin{aligned}
\int_{\nu}^{\infty}\mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma &\geq \int_{\nu}^{\infty}\mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))}\mathrm{d}\varsigma \\
&\geq \frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))}\int_{\nu}^{\infty}\mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \\
&\geq \frac{\omega^{\alpha}(\nu)}{\mathcal{R}^{\alpha}(\varrho(\nu,b))}\int_{\nu}^{\infty}\mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma. \qquad (2.19)
\end{aligned}$$

Using (2.18) and (2.19) in (2.17) yields

$$\omega^{1-\alpha}(\nu) \geq \mathcal{R}(\nu) \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma + \frac{1}{\mathcal{R}^{\alpha}(\varrho(\nu,b))} \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma.$$

Taking the lim sup at $\nu \to \infty$ of the resulting inequality, we obtain

$$\limsup_{\nu\to\infty} \left(\mathcal{R}(\nu) \int_{\nu_0}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma + \mathcal{R}^{-\alpha}(\varrho(\nu,b)) \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \right) \leq \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases}$$

which contradicts condition (2.2). Consequently, the proof is concluded. □

Theorem 2.2. *Let* (1.2) *and* (2.1) *hold. If*

$$\limsup_{\nu\to\infty} \left(\mathcal{R}(\varrho(\nu,b)) \int_{\nu_0}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma + \int_{\varrho(\nu,b)}^{\nu} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma \right.$$
$$\left. + \mathcal{R}^{-\alpha}(\varrho(\nu,b)) \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \right) > \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases} \tag{2.20}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $\varkappa(\nu)$ be a nonoscillatory solution of Eq. (1.1), say $\varkappa(\nu) > 0$, $\varkappa(\eta(\nu)) > 0$, and $\varkappa(\varrho(\nu,b)) > 0$ for $\nu \geq \nu_1$ for some $\nu_1 \geq \nu_0$. By proceeding analogously to the proof of Theorem 2.1, we derive (2.17) as follows

$$\omega(\nu) \geq \mathcal{R}(\nu) \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma + \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma,$$

which can equivalently be expressed as

$$\omega(\varrho(\nu,b)) \geq \mathcal{R}(\varrho(\nu,b)) \int_{\nu_5}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma + \int_{\varrho(\nu,b)}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma,$$

or equivalently,

$$\begin{aligned} \omega(\varrho(\nu,b)) \geq\; & \mathcal{R}(\varrho(\nu,b)) \int_{\nu_5}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma + \int_{\varrho(\nu,b)}^{\nu} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \\ & + \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma. \end{aligned} \tag{2.21}$$

Since ω is decreasing, it follows that

$$\int_{\nu_5}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \geq \omega^{\alpha}(\varrho(\nu,b)) \int_{\nu_5}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma, \tag{2.22}$$

and

$$\int_{\varrho(\nu,b)}^{\nu} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \geq \omega^{\alpha}(\varrho(\nu,b)) \int_{\varrho(\nu,b)}^{\nu} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma. \tag{2.23}$$

Since $\omega(\nu)/\mathcal{R}(\nu)$ increasing, then

$$\begin{aligned} \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma &\geq \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))}\mathrm{d}\varsigma \\ &\geq \frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))} \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma. \end{aligned} \tag{2.24}$$

By substituting (2.22), (2.23) and (2.24) into (2.21), we have

$$\begin{aligned}\omega\left(\varrho\left(\nu,\ell\right)\right) &\geq \mathcal{R}\left(\varrho\left(\nu,b\right)\right)\omega^{\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\nu_5}^{\varrho(\nu,b)}\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma+\omega^{\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\varrho(\nu,b)}^{\nu}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma \\ &\quad+\frac{\omega^{\alpha}\left(\varrho\left(\nu,b\right)\right)}{\mathcal{R}^{\alpha}\left(\varrho\left(\nu,b\right)\right)}\int_{\nu}^{\infty}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathcal{R}^{\alpha}\left(\varrho\left(\varsigma,b\right)\right)\mathrm{d}\varsigma.\end{aligned}$$

which can be written

$$\begin{aligned}\omega^{1-\alpha}\left(\varrho\left(\nu,b\right)\right) &\geq \mathcal{R}\left(\varrho\left(\nu,b\right)\right)\int_{\nu_5}^{\varrho(\nu,b)}\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma+\int_{\varrho(\nu,b)}^{\nu}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma \\ &\quad+\mathcal{R}^{-\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\nu}^{\infty}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathcal{R}^{\alpha}\left(\varrho\left(\varsigma,b\right)\right)\mathrm{d}\varsigma.\end{aligned}$$

By taking $\lim\limits_{\nu\to\infty}$ sup of the resulting inequality, we obtain

$$\begin{aligned}&\limsup_{\nu\to\infty}\Bigg(\mathcal{R}\left(\varrho\left(\nu,b\right)\right)\int_{\nu_0}^{\varrho(\nu,b)}\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma+\int_{\varrho(\nu,b)}^{\nu}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma \\ &\quad+\mathcal{R}^{-\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\nu}^{\infty}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathcal{R}^{\alpha}\left(\varrho\left(\varsigma,b\right)\right)\mathrm{d}\varsigma\Bigg)\leq\left\{\begin{array}{ll}1 & \text{if } \alpha=1,\\ 0 & \text{if } 0<\alpha<1.\end{array}\right.\end{aligned}$$

a contradiction to (2.20). The proof of this theorem is complete. □

Theorem 2.3. *Let* (1.2) *and* (2.1) *hold. If*

$$\limsup_{\nu\to\infty}\int_{\varrho(\nu,b)}^{\nu}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma>\left\{\begin{array}{ll}1 & \text{if } \alpha=1,\\ 0 & \text{if } 0<\alpha<1,\end{array}\right. \tag{2.25}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $\varkappa\left(\nu\right)$ be a nonoscillatory solution of equation (1.1), with $\varkappa\left(\nu\right)>0$, $\varkappa\left(\eta\left(\nu\right)\right)>0$, and $\varkappa\left(\varrho\left(\nu,b\right)\right)>0$ for $\nu\geq\nu_1\geq\nu_0$. By proceeding analogously to the proof of Theorem 2.1, we again obtain inequalities (2.11) and (2.14). Define

$$y\left(\nu\right):=\mathcal{R}\left(\nu\right)\kappa(\nu)\omega'(\nu)+\omega\left(\nu\right).$$

Since $\omega'(\nu)<0$ under the case (N_2), it follows that $0\leq y\left(\nu\right)\leq\omega\left(\nu\right)$. Consequently, inequality (2.14) can be expressed as

$$y'\left(\nu\right)+\mathcal{R}\left(\nu\right)\widehat{\mathfrak{q}}\left(\nu\right)y^{\alpha}\left(\varrho\left(\nu,b\right)\right)\leq 0.$$

Integrate this inequality from $\varrho\left(\nu,b\right)$ to ν, we get

$$\begin{aligned}y\left(\varrho\left(\nu,b\right)\right) &\geq \int_{\varrho(\nu,b)}^{\nu}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)y^{\alpha}\left(\varrho\left(\varsigma,b\right)\right)\mathrm{d}\varsigma, \\ &\geq y^{\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\varrho(\nu,b)}^{\nu}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma.\end{aligned}$$

Equivalently, this gives

$$y^{1-\alpha}\left(\varrho\left(\nu,b\right)\right)\geq\int_{\varrho(\nu,b)}^{\nu}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma.$$

Taking the lim sup and noting that $\omega(\nu) \to 0$, which implies that $y(\nu) \to 0$, we arrive at

$$\limsup_{\nu\to\infty} \int_{\varrho(\nu,b)}^{\nu} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma \le \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases}$$

which contradicts (2.25). Hence, the proof of the theorem is complete. □

Remark 2.2. *Subsequently, two oscillation criteria for Eq.* (1.1) *are introduced for the scenario where* $\varrho(\nu,b)$ *denotes an advanced argument, i.e.,* $\varrho(\nu,b) \ge \nu$.

Theorem 2.4. *Let* (1.2) *and* (2.1) *hold. If*

$$\begin{aligned} &\limsup_{\nu\to\infty} \Bigg(\mathcal{R}^{\alpha}(\varrho(\nu,b))\,\mathcal{R}^{1-\alpha}(\nu) \int_{\nu_0}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma \\ &\quad + \mathcal{R}^{-\alpha}(\nu) \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \Bigg) > \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases} \end{aligned} \tag{2.26}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $\varkappa(\nu)$ be a nonoscillatory solution of Eq. (1.1), say $\varkappa(\nu) > 0$, $\varkappa(\eta(\nu)) > 0$, and $\varkappa(\varrho(\nu,b)) > 0$ for $\nu \ge \nu_1$ for some $\nu_1 \ge \nu_0$. By proceeding analogously to the proof of Theorem 2.1, we again obtain (2.17). Using the fact that $\omega(\nu)$ is decreasing while $\omega(\nu)/\mathcal{R}(\nu)$ is increasing, we note that for $\varsigma \le \nu$, it holds that $\varsigma \le \varrho(\varsigma,b) \le \varrho(\nu,b)$ and $\varrho(\nu,b) \ge \nu$. Hence,

$$\omega(\varrho(\varsigma,b)) \ge \omega(\varrho(\nu,b)).$$

Consequently,

$$\begin{aligned} \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma &\ge \mathcal{R}^{\alpha}(\varrho(\nu,b)) \frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))} \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma \\ &\ge \frac{\mathcal{R}^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\nu)} \omega^{\alpha}(\nu) \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma. \end{aligned} \tag{2.27}$$

Similarly, for $\varsigma \ge \nu$, we have $\varrho(\varsigma,b) \ge \varrho(\nu,b)$, which implies

$$\frac{\omega^{\alpha}(\varrho(\varsigma,b))}{\mathcal{R}^{\alpha}(\varrho(\varsigma,b))} \ge \frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))} \ge \frac{\omega^{\alpha}(\nu)}{\mathcal{R}^{\alpha}(\nu)}.$$

Therefore,

$$\begin{aligned} \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma &\ge \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b)) \frac{\omega^{\alpha}(\nu)}{\mathcal{R}^{\alpha}(\nu)}\mathrm{d}\varsigma \\ &\ge \frac{\omega^{\alpha}(\nu)}{\mathcal{R}^{\alpha}(\nu)} \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma. \end{aligned} \tag{2.28}$$

Using (2.27) and (2.28) in (2.17) yields

$$\begin{aligned} \omega(\nu) &\ge \mathcal{R}(\nu) \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma + \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\omega^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma \\ &\ge \mathcal{R}(\nu) \frac{\mathcal{R}^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\nu)} \omega^{\alpha}(\nu) \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma + \frac{\omega^{\alpha}(\nu)}{\mathcal{R}^{\alpha}(\nu)} \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\,\widehat{\mathfrak{q}}(\varsigma)\,\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\,\mathrm{d}\varsigma, \end{aligned}$$

or equivalently

$$\omega^{1-\alpha}(\nu) \geq \frac{\mathcal{R}^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha-1}(\nu)} \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\, d\varsigma + \frac{1}{\mathcal{R}^{\alpha}(\nu)} \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\widehat{\mathfrak{q}}(\varsigma)\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\, d\varsigma,$$

we taking lim sup of the resulting inequality, we obtain

$$\begin{aligned}\limsup_{\nu\to\infty} \Bigg(& \frac{\mathcal{R}^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha-1}(\nu)} \int_{\nu_0}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\, d\varsigma \\ & + \frac{1}{\mathcal{R}^{\alpha}(\nu)} \int_{\nu}^{\infty} \mathcal{R}(\varsigma)\widehat{\mathfrak{q}}(\varsigma)\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\, d\varsigma \Bigg) \leq \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases}\end{aligned}$$

which contradicts (2.26). Therefore, the proof is complete. □

Theorem 2.5. *Let (1.2) and (2.1) hold. If*

$$\begin{aligned}\limsup_{\nu\to\infty} \Bigg(& \mathcal{R}(\varrho(\nu,b)) \int_{\nu_0}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\, d\varsigma + \mathcal{R}^{1-\alpha}(\varrho(\nu,b)) \int_{\nu}^{\varrho(\nu,b)} \mathcal{R}^{\alpha}(\varrho(\varsigma,b))\widehat{\mathfrak{q}}(\varsigma)\, d\varsigma \\ & + \mathcal{R}^{-\alpha}(\varrho(\nu,b)) \int_{\varrho(\nu,b)}^{\infty} \mathcal{R}(\varsigma)\widehat{\mathfrak{q}}(\varsigma)\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\, d\varsigma \Bigg) > \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases}\end{aligned} \tag{2.29}$$

then Eq. (1.1) is oscillatory.

Proof. Let $\varkappa(\nu)$ be a nonoscillatory solution of Eq. (1.1), with $\varkappa(\nu) > 0$, $\varkappa(\eta(\nu)) > 0$, and $\varkappa(\varrho(\nu,b)) > 0$ for $\nu \geq \nu_1 \geq \nu_0$. By proceeding analogously to the proof of Theorem 2.1, we again obtain (2.17), which can be expressed as

$$\begin{aligned}\omega(\varrho(\nu,b)) \geq\ & \mathcal{R}(\varrho(\nu,b)) \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\omega^{\alpha}(\varrho(\varsigma,b))\, d\varsigma + \mathcal{R}(\varrho(\nu,b)) \int_{\nu}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\omega^{\alpha}(\varrho(\varsigma,b))\, d\varsigma \\ & + \int_{\varrho(\nu,b)}^{\infty} \mathcal{R}(\varsigma)\widehat{\mathfrak{q}}(\varsigma)\omega^{\alpha}(\varrho(\varsigma,b))\, d\varsigma.\end{aligned} \tag{2.30}$$

Using the monotonicity of ω and $\omega/\mathcal{R}$, we obtain

$$\int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\omega^{\alpha}(\varrho(\varsigma,b))\, d\varsigma \geq \omega^{\alpha}(\varrho(\nu,b)) \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\, d\varsigma,$$

$$\int_{\nu}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\omega^{\alpha}(\varrho(\varsigma,b))\, d\varsigma \geq \frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))} \int_{\nu}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\, d\varsigma,$$

and

$$\int_{\varrho(\nu,b)}^{\infty} \mathcal{R}(\varsigma)\widehat{\mathfrak{q}}(\varsigma)\omega^{\alpha}(\varrho(\varsigma,b))\, d\varsigma \geq \frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))} \int_{\varrho(\nu,b)}^{\infty} \mathcal{R}(\varsigma)\widehat{\mathfrak{q}}(\varsigma)\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\, d\varsigma.$$

Substituting these inequalities into (2.30), we obtain

$$\begin{aligned}\omega(\varrho(\nu,b)) \geq\ & \mathcal{R}(\varrho(\nu,b))\omega^{\alpha}(\varrho(\nu,b)) \int_{\nu_5}^{\nu} \widehat{\mathfrak{q}}(\varsigma)\, d\varsigma \\ & + \mathcal{R}^{1-\alpha}(\varrho(\nu,b))\omega^{\alpha}(\varrho(\nu,b)) \int_{\nu}^{\varrho(\nu,b)} \widehat{\mathfrak{q}}(\varsigma)\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\, d\varsigma \\ & + \frac{\omega^{\alpha}(\varrho(\nu,b))}{\mathcal{R}^{\alpha}(\varrho(\nu,b))} \int_{\varrho(\nu,b)}^{\infty} \mathcal{R}(\varsigma)\widehat{\mathfrak{q}}(\varsigma)\mathcal{R}^{\alpha}(\varrho(\varsigma,b))\, d\varsigma,\end{aligned}$$

or equivalently,

$$\begin{aligned} \frac{\omega\left(\varrho\left(\nu,b\right)\right)}{\omega^{\alpha}\left(\varrho\left(\nu,b\right)\right)} \geq\ & \mathcal{R}\left(\varrho\left(\nu,b\right)\right)\int_{\nu_5}^{\nu}\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma+\mathcal{R}^{1-\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\nu}^{\varrho(\nu,b)}\widehat{\mathfrak{q}}\left(\varsigma\right)\mathcal{R}^{\alpha}\left(\varrho\left(\varsigma,b\right)\right)\mathrm{d}\varsigma \\ & +\mathcal{R}^{-\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\varrho(\nu,b)}^{\infty}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathcal{R}^{\alpha}\left(\varrho\left(\varsigma,b\right)\right)\mathrm{d}\varsigma. \end{aligned}$$

Taking the lim sup yields

$$\begin{aligned} \limsup_{\nu\to\infty}\Bigg(&\mathcal{R}\left(\varrho\left(\nu,b\right)\right)\int_{\nu_0}^{\nu}\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma+\mathcal{R}^{1-\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\nu}^{\varrho(\nu,b)}\mathcal{R}^{\alpha}\left(\varrho\left(\varsigma,b\right)\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma \\ &+\mathcal{R}^{-\alpha}\left(\varrho\left(\nu,b\right)\right)\int_{\varrho(\nu,b)}^{\infty}\mathcal{R}\left(\varsigma\right)\widehat{\mathfrak{q}}\left(\varsigma\right)\mathcal{R}^{\alpha}\left(\varrho\left(\varsigma,b\right)\right)\mathrm{d}\varsigma\Bigg) \leq \begin{cases} 1 & \text{if } \alpha=1, \\ 0 & \text{if } 0<\alpha<1, \end{cases} \end{aligned}$$

which contradicts (2.29). Hence, the proof is complete. □

3 Examples

This section provides illustrative examples that serve to demonstrate the significance and applicability of the main results.

Example 3.1. *Consider the equation*

$$\left(\nu^{2}\left(\varkappa\left(\nu\right)+\frac{1}{5}\varkappa\left(\frac{\nu}{4}\right)\right)'\right)'+\int_{0}^{1}\mathfrak{q}_0\varkappa\left(\frac{\ell}{2}\nu\right)\mathrm{d}\ell=0,\nu\geq 1. \tag{3.1}$$

Here $\kappa\left(\nu\right)=\nu^{2}$, $\mathfrak{p}\left(\nu\right)=\frac{1}{5}$, $\eta\left(\nu\right)=\frac{\nu}{4}$, $\mathfrak{q}\left(\nu,\ell\right)=\mathfrak{q}_0>0$, $\varrho\left(\nu,b\right)=\frac{\nu}{2}$, $\alpha=1$, $a=0$, $b=1$. *Since*

$$\mathcal{R}(\nu)=\int_{\nu}^{\infty}\frac{1}{\varsigma^{2}}\mathrm{d}\varsigma=\frac{1}{\nu},$$

we have

$$\mathcal{R}\left(\varrho\left(\nu,b\right)\right)=\mathcal{R}(\frac{\nu}{2})=\frac{2}{\nu},\quad \mathcal{R}\left(\eta\left(\varrho\left(\nu,b\right)\right)\right)=\mathcal{R}(\frac{\nu}{8})=\frac{8}{\nu},\quad \mathfrak{p}\left(\varrho\left(\nu,b\right)\right)=\frac{1}{5},\ \widetilde{\mathfrak{q}}\left(\nu\right)=\mathfrak{q}_0.$$

Moreover,

$$\begin{aligned} \widehat{\mathfrak{q}}\left(\nu\right) &= \int_{0}^{1}\mathfrak{q}_0\left(1-\frac{1}{5}\cdot\frac{\mathcal{R}\left(\left(\ell/8\right)\nu\right)}{\mathcal{R}\left(\left(\ell/2\right)\nu\right)}\right)\mathrm{d}\ell \\ &= \int_{0}^{1}\mathfrak{q}_0\left(1-\frac{1}{5}\cdot\frac{8/\left(\ell\nu\right)}{2/\left(\ell\nu\right)}\right)\mathrm{d}\ell \\ &= \int_{0}^{1}\mathfrak{q}_0\left(1-\frac{4}{5}\right)\mathrm{d}\ell \\ &= \int_{0}^{1}\frac{1}{5}\mathfrak{q}_0\mathrm{d}\ell=0.2q_0. \end{aligned}$$

Step 1: *Condition* (2.1)

$$\int_{\nu_0}^{\infty}\mathcal{R}\left(\varsigma\right)\widetilde{\mathfrak{q}}\left(\varsigma\right)\mathrm{d}\varsigma=\int_{1}^{\infty}\frac{\mathfrak{q}_0}{\varsigma}\mathrm{d}\varsigma=\infty,$$

hence (2.1) *is satisfied.*
Step 2: *Condition* (2.2)

$$\begin{aligned}
&\limsup_{\nu\to\infty}\left(\frac{1}{\nu}\int_1^{\nu}0.2q_0\mathrm{d}\varsigma+\frac{\nu}{2}\int_{\nu}^{\infty}\frac{1}{\varsigma}\cdot 0.2q_0\cdot\frac{2}{\varsigma}\mathrm{d}\varsigma\right)\\
&\quad=\limsup_{\nu\to\infty}\left(\frac{1}{\nu}\cdot 0.2q_0\,(\nu-1)+\frac{\nu}{2}\cdot 0.4q_0\frac{1}{\nu}\right)\\
&\quad=\limsup_{\nu\to\infty}\left(0.2q_0\left(1-\frac{1}{\nu}\right)+0.2q_0\right)\\
&\quad=0.4q_0.
\end{aligned}$$

Thus (2.2) *holds if*

$$0.4q_0>1\iff \mathfrak{q}_0>2.5. \tag{3.2}$$

Step 3: *Condition* (2.20)

$$\begin{aligned}
&\limsup_{\nu\to\infty}\left(\mathcal{R}\left(\frac{\nu}{2}\right)\int_1^{\frac{\nu}{2}}0.2q_0\mathrm{d}\varsigma+\int_{\frac{\nu}{2}}^{\nu}\frac{1}{\varsigma}\cdot 0.2q_0\mathrm{d}\varsigma+\mathcal{R}^{-1}\left(\frac{\nu}{2}\right)\int_{\nu}^{\infty}\frac{1}{\varsigma}\cdot 0.2q_0\cdot\mathcal{R}\left(\frac{\varsigma}{2}\right)\mathrm{d}\varsigma\right)\\
&=\limsup_{\nu\to\infty}\left(\frac{2}{\nu}\cdot 0.2q_0\left(\frac{\nu}{2}-1\right)+0.2q_0\ln 2+0.2q_0\right)\\
&=\limsup_{\nu\to\infty}\left(0.2q_0-\frac{0.4q_0}{\nu}+0.2q_0\ln 2+0.2q_0\right)\\
&=(0.4+0.2\,\ln 2)\,\mathfrak{q}_0.
\end{aligned}$$

Therefore (2.20) *is satisfied whenever*

$$\mathfrak{q}_0>\frac{1}{(0.4+0.2\ln 2)}\simeq 1.8566. \tag{3.3}$$

Step 4: *Condition* (2.25)

$$\limsup_{\nu\to\infty}\int_{\frac{\nu}{2}}^{\nu}\frac{1}{\varsigma}0.2q_0\mathrm{d}\varsigma=0.2q_0\ln 2.$$

Hence (2.25) *holds provided that*

$$\mathfrak{q}_0>\frac{5}{\ln 2}\simeq 7.2135. \tag{3.4}$$

Applying Theorems 2.1*-*2.3 *to Eq.* (3.1)*, we obtain:*

(*i*) *By Theorem* 2.1*, oscillation occurs if condition* (3.2) *holds, i.e.,* $\mathfrak{q}_0>2.5$.

(*ii*) *By Theorem* 2.2*, oscillation occurs if condition* (3.3) *holds, i.e.,* $\mathfrak{q}_0>1.856$.

(*iii*) *By Theorem* 2.3*, oscillation occurs if condition* (3.4) *holds, i.e.,* $\mathfrak{q}_0>7.2135$.

In particular, the critical value $\mathfrak{q}_0>7.2135$ *ensures that the oscillation criteria stated in Theorems* 2.1*-*2.3 *are satisfied simultaneously. Therefore, Eq.* (3.1) *is oscillatory whenever* $\mathfrak{q}_0>7.2135$.

Note that the results in [20] *are not applicable to Eq.* (3.1)*; thus, the findings of the present study extend and improve upon these earlier works.*

Example 3.2. *Consider the equation*

$$\left(\frac{\nu^3}{2}\left(\varkappa(\nu)+\frac{1}{11}\varkappa\left(\frac{\nu}{3}\right)\right)'\right)'+\int_0^1 \nu x^{1/3}(3\nu\ell)\,\mathrm{d}\ell=0, \nu\geq 1. \tag{3.5}$$

Here we have: $\alpha=\frac{1}{3}$, $a=0$, $b=1$, $\kappa(\nu)=\frac{\nu^3}{2}$, $\mathfrak{p}(\nu)=\frac{1}{11}$, $\eta(\nu)=\frac{\nu}{3}$, $\mathfrak{q}(\nu,1)=\nu$, $\varrho(\nu,1)=3\nu$. *Then*

$$\mathcal{R}(\nu)=\int_\nu^\infty \frac{2}{\varsigma^3}\mathrm{d}\varsigma=\frac{1}{\nu^2},\ \mathcal{R}(1)=1,$$

$$\mathcal{R}(\varrho(\nu,1))=\mathcal{R}(3\nu)=\frac{1}{(3\nu)^2}=\frac{1}{9\nu^2},$$

$$\mathcal{R}(\eta(\varrho(\nu,1)))=\mathcal{R}\left(\frac{1}{3}(3\nu)\right)=\mathcal{R}(\nu)=\frac{1}{\nu^2},$$

and

$$\widetilde{\mathfrak{q}}(\nu)=\int_0^1 \nu\,\mathrm{d}\ell=\nu.$$

Hence

$$\widehat{\mathfrak{q}}(\nu)=\int_0^1 \nu\left(1-\frac{1}{11}\frac{1/\left(\nu^2\ell^2\right)}{1/\left(9\nu^2\ell^2\right)}\right)^{\frac{1}{3}}\mathrm{d}\ell=\int_0^1 \nu\left(1-\frac{9}{11}\right)^{\frac{1}{3}}\mathrm{d}\ell=\nu\left(\frac{2}{11}\right)^{\frac{1}{3}}\simeq 0.567\nu.$$

Moreover,

$$\int_{\nu_0}^\infty \mathcal{R}(\varsigma)\,\widetilde{\mathfrak{q}}(\varsigma)\,\mathrm{d}\varsigma=\int_1^\infty \frac{1}{\varsigma^2}\cdot\varsigma\mathrm{d}\varsigma=\int_1^\infty \frac{1}{\varsigma}\mathrm{d}\varsigma=\infty.$$

Applying condition (2.26), *we obtain*

$$\begin{aligned}
&\limsup_{\nu\to\infty}\left(\frac{\mathcal{R}^{\frac{1}{3}}(3\nu)}{\mathcal{R}^{\frac{1}{3}-1}(\nu)}\int_1^\nu 0.567\varsigma\mathrm{d}\varsigma+\frac{1}{\mathcal{R}^{\frac{1}{3}}(\nu)}\int_\nu^\infty \frac{1}{\varsigma^2}(0.567\varsigma)\,\mathcal{R}^{\frac{1}{3}}(3\varsigma)\,\mathrm{d}\varsigma\right)\\
&=\limsup_{\nu\to\infty}\left(\frac{1}{9^{\frac{1}{3}}}\cdot 0.283\left(1-\frac{1}{\nu^2}\right)+\frac{0.567}{9^{\frac{1}{3}}}\nu^{\frac{2}{3}}\cdot\frac{3}{2}\frac{1}{\nu^{\frac{2}{3}}}\right)\\
&=\limsup_{\nu\to\infty}\left(0.136\cdot\left(1-\frac{1}{\nu^2}\right)+0.409\right)\\
&=0.545>0.
\end{aligned}$$

Thus, condition (2.26) *of Theorem* 2.4 *is satisfied. Next, consider condition* (2.29):

$$\begin{aligned}
&\limsup_{\nu\to\infty}\left(\mathcal{R}(3\nu)\int_1^\nu 0.567\varsigma\mathrm{d}\varsigma+\mathcal{R}^{2/3}(3\nu)\int_\nu^{3\nu}\mathcal{R}^{1/3}(3\varsigma)\,(0.567\varsigma)\,\mathrm{d}\varsigma\right.\\
&\left.+\mathcal{R}^{-1/3}(3\nu)\int_{3\nu}^\infty \mathcal{R}(\varsigma)\,(0.567\varsigma)\,\mathcal{R}^{1/3}(3\varsigma)\,\mathrm{d}\varsigma\right.\\
&=\limsup_{\nu\to\infty}\left(\frac{1}{9\nu^2}\cdot\frac{0.567}{2}\left(\nu^2-1\right)+\frac{1}{9^{2/3}\nu^{4/3}}\cdot\frac{0.567}{9^{1/3}}\frac{3}{4}\left(3^{4/3}-1\right)\nu^{4/3}\right.\\
&\left.+9^{1/3}\nu^{2/3}\cdot\frac{0.567}{9^{1/3}}\cdot\frac{3}{2}\frac{1}{(3\nu)^{2/3}}\right)\\
&=\limsup_{\nu\to\infty}\left(0.0315\left(1-\frac{1}{\nu^2}\right)+0.157+0.409\right)\\
&\simeq 0.597>0.
\end{aligned}$$

Therefore, condition (2.29) of Theorem 2.5 also holds. As the conditions of Theorems 2.4 and 2.5 are fulfilled, it follows that Eq. (3.5) is oscillatory. It is worth noting that the results in [20] do not apply to Eq. (3.5); therefore, this study provides an extension and improvement of the existing literature.

4 Conclusions

This study focused on analyzing the oscillatory behavior of solutions to a class of second-order nonlinear neutral differential equations, addressing both delays and advanced time arguments. The results demonstrated that sufficient conditions can be defined to ensure the oscillation of all solutions, enhancing a deeper understanding of the dynamical properties of these complex equations. The results also extend previous studies by introducing more flexible and less restrictive criteria and including equations with delays and advanced time arguments, enabling their application in a variety of practical and theoretical contexts. It is interesting for future studies to extend the results to higher-order neutral differential equations

$$\left(\kappa(\nu)\omega^{(n-1)}(\nu)\right)' + \int_a^b \mathfrak{q}(\nu,\ell)\,\varkappa^{\alpha}(\varrho(\nu,\ell))\,\mathrm{d}\ell = 0,\ n \geq 4,$$

or second-order equations containing a superlinear neutral term

$$\left(\kappa(\nu)\left(\varkappa(\nu) + \mathfrak{p}(\nu)\,\varkappa^{\gamma}(\eta(\nu))\right)'\right)' + \int_a^b \mathfrak{q}(\nu,\ell)\,\varkappa^{\alpha}(\varrho(\nu,\ell))\,\mathrm{d}\ell = 0.$$

This opens up new avenues for practical applications and explores deeper dynamical properties of this class of equations.

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