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Survey Article

Positive Linear Framework for Neural Networks with Generalized Logistic Structure: Multivariate and Finite Domain Case

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Abstract: In this work, we examine a family of abstract neural network operators in a multivariate setting, defined on compact subsets of $\mathbb{R}^N$ and taking values in general Banach spaces. These operators are constructed using a symmetric density derived from a modified logistic-type activation function with adjustable parameters. Within the framework of positive linear operator theory, we provide quantitative convergence estimates towards the identity operator. Our results are formulated using tools such as the modulus of continuity and Fréchet derivatives. Furthermore, we offer a detailed discussion on the behavior of the operators when applied to convex functions. This article reflects both the author's contributions to the field over the past 30 years and some recent developments in neural network approximation theory.

Keywords: Neural network operators; positive linear operators; multivariate-abstract quantitative approximation; convexity; generalized logistic activation function.

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1 Introduction

Since 1985, the author has been actively involved in the quantitative analysis of how positive linear operators approximate the identity operator; for background, see [1–7]. His earlier contributions were grounded in studying the weak convergence of finite positive measures toward the Dirac delta, employing geometric moment techniques (cf. [2]). These efforts resulted in optimal upper estimates, including sharp Jackson-type inequalities (see [1], [2]). Over the years, the author's investigations have evolved across both univariate and multivariate frameworks. Furthermore, since 1997, the author has also developed a line of research devoted to analyzing the convergence behavior of neural network operators. This has led to a considerable number of scholarly outputs – articles as well as monographs (e.g., [3], [5–7]) – which form the theoretical basis for the present work. The various neural network operators examined in those studies are inherently positive linear in nature.

In this study, multivariate-abstract neural network operators are for the first time treated explicitly within the framework of positive linear operators. As a consequence, established techniques from the theory of positive linear mappings can now be applied to these finite-sum-based neural operators,

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yielding novel and meaningful results, both in terms of pointwise behavior and uniform convergence. The application of the Riesz representation theorem links the neural operator construction to measure-theoretic principles. Moreover, when the target function possesses convexity, one can derive even sharper and more refined bounds.

The adoption of a perturbed generalized logistic function as the activation mechanism is thoroughly supported by the literature, especially in [6] (see Chapters 15 and 16). The generalized logistic function itself is known for its robust and effective behavior and remains one of the most widely employed activation functions in this field. The symmetrization method introduced in this article is designed to enhance the speed of convergence of the proposed operators toward the identity, even when only partial data input is utilized.

This article introduces, for the first time, a theoretical link between two major fields: neural network approximation, which is an essential component of artificial intelligence, and the theory of positive linear operators, a fundamental area within functional analysis. Due to its theoretical nature, the work primarily targets experts familiar with these domains. The author acknowledges the significant influence of classical contributions such as [13] and [14]. For foundational insights into neural networks, readers may consult [9–11] and [15–17], while for more contemporary studies, references [18–27] provide valuable developments.

2 Basics

We begin by referring to the framework outlined in [6, pp. 395–471]. In our case, the activation mechanism is defined through a generalized logistic-type expression that depends on the parameters q and λ as follows:

$$\varphi_{q,\lambda}(x) = \frac{1}{1+qA^{-\lambda x}}, \quad x \in \mathbb{R}, \ q, \lambda > 0, \ A > 1,$$

where the function represents an A-extended version of the classical logistic function. For more read Chapter 16 of [6]: "Banach space valued ordinary and fractional neural network approximation based on q-deformed and λ-parametrized A-generalized logistic function". Our present study is inspired in part by the material presented in Chapters 15 and 16 of [6].

In our construction, we introduce a "symmetrization method" designed to operate with a reduced data input, effectively utilizing only half of the available feed for the neural network. To implement this, we define the density function as

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}, \ q, \lambda > 0,$$

which plays a central role in the formulation of our operators. We observe the following properties:

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \tag{2.1}$$

and

$$G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x), \ \forall\, x \in \mathbb{R}. \tag{2.2}$$

Adding (2.1) and (2.2) we obtain

$$G_{q,\lambda}(-x) + G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x), \ \forall\, x \in \mathbb{R},$$

which gives that

$$W(x) := \frac{G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x)}{2}$$

is an even function. Also this function is symmetric with respect to the y-axis. The equality (16.18) of [6, p. 401] shows

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^{\lambda}-1}{2\left(A^{\lambda}+1\right)},$$

which corresponds to the global max of $G_{q,\lambda}$. Similarly we get

$$G_{\frac{1}{q},\lambda}\left(\frac{\log_A \frac{1}{q}}{\lambda}\right) = G_{\frac{1}{q},\lambda}\left(\frac{-\log_A q}{\lambda}\right) = \frac{A^{\lambda}-1}{2\left(A^{\lambda}+1\right)},$$

which is the global max of $G_{\frac{1}{q},\lambda}$.

Theorem 16.1 of [6, p. 401] gives

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(x-i\right) = 1, \ \ \forall\, x \in \mathbb{R}, \ \lambda, q > 0, \ \ A > 1,$$

and

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}\left(x-i\right) = 1, \ \ \forall\, x \in \mathbb{R}, \ \lambda, q > 0, \ \ A > 1.$$

Hence, we get

$$\sum_{i=-\infty}^{\infty} W\left(x-i\right) = 1, \ \ \forall\, x \in \mathbb{R}.$$

Theorem 16.2 of [6, p. 402] implies

$$\int_{-\infty}^{\infty} G_{q,\lambda}\left(x\right) dx = 1, \ \ \ \lambda, q > 0, \ \ A > 1,$$

and hence

$$\int_{-\infty}^{\infty} G_{\frac{1}{q},\lambda}\left(x\right) dx = 1,$$

so that

$$\int_{-\infty}^{\infty} W\left(x\right) dx = 1,$$

i.e., W is a density function.

We also get the next result from Theorem 16.3 of [6, p. 402]:

"Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ \left|nx-k\right| \geq n^{1-\alpha} \end{array}\right.}^{\infty} G_{q,\lambda}\left(nx-k\right) < 2\max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda\left(n^{1-\alpha}-2\right)}} = \widetilde{\gamma} A^{-\lambda\left(n^{1-\alpha}-2\right)},$$

where $\lambda, q > 0, \ A > 1$; $\widetilde{\gamma} := 2\max\left\{q, \frac{1}{q}\right\}$."

Similarly, we observe

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ \left|nx-k\right| \geq n^{1-\alpha} \end{array}\right.}^{\infty} G_{\frac{1}{q},\lambda}\left(nx-k\right) < \widetilde{\gamma} A^{-\lambda\left(n^{1-\alpha}-2\right)}.$$

Therefore, we derive

$$\sum_{\begin{cases} k=-\infty \\ |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} W(nx-k) < \widetilde{\gamma} A^{-\lambda\left(n^{1-\alpha}-2\right)},$$

where $\widetilde{\gamma} := 2\max\left\{q, \frac{1}{q}\right\}$. As usual, $\lceil\cdot\rceil$ represents the smallest integer greater than or equal to the given number, while $\lfloor\cdot\rfloor$ denotes its greatest integer less than or equal to it.

By Theorem 16.4 of [6, p. 402] we have:

"Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. For $q > 0$, $\lambda > 0$, $A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, and $\lambda_q > 1$. Then

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)} < \max\left\{\frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: K(q)."$$

holds." Similarly, we consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Thus

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}(nx-k)} < \max\left\{\frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \frac{1}{G_{q,\lambda}(\lambda_q)}\right\} = K(q).$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k) > \frac{1}{K(q)},$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{\frac{1}{q},\lambda}(nx-k) > \frac{1}{K(q)}.$$

Consequently it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(G_{q,\lambda}(nx-k) + G_{\frac{1}{q},\lambda}(nx-k)\right)}{2} > \frac{2}{2K(q)} = \frac{1}{K(q)},$$

so that

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(G_{q,\lambda}(nx-k)+G_{\frac{1}{q},\lambda}(nx-k)\right)}{2}} < K(q),$$

that is

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} W(nx-k)} < K(q).$$

So, we have proved

Theorem 2.1. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. For $q, \lambda > 0$, $A > 1$, we consider $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, and $\lambda_q > 1$. Also consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} W(nx-k)} < K(q).$$

We state

Remark 2.1. *(I) Remark* 16.5 *of* [6, p. 402] *implies that, for some* $x_1, x_2 \in [a, b]$,

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx_1 - k) \neq 1,$$

and

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx_2 - k) \neq 1$$

hold, which gives

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(G_{q,\lambda}(nx_1 - k) + G_{\frac{1}{q},\lambda}(nx_2 - k)\right)}{2} \neq 1.$$

Hence

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(G_{q,\lambda}(nx_1 - k) + G_{\frac{1}{q},\lambda}(nx_1 - k)\right)}{2} \neq 1,$$

even if

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx_1 - k) = 1,$$

because then

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{G_{q,\lambda}(nx_1 - k)}{2} + \frac{1}{2} \neq 1,$$

equivalently

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{G_{q,\lambda}(nx_1 - k)}{2} \neq \frac{1}{2},$$

true by

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx_1 - k) \neq 1.$$

(II) For a given interval $[a, b] \subset \mathbb{R}$ *and a large number* n*, we assume* $\lceil na \rceil \leq \lfloor nb \rfloor$*. Also* $a \leq \frac{k}{n} \leq b$*, iff* $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$*. Then, we get*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \leq 1.$$

We now turn our attention to the multivariate setting, taking Chapter 15 of [6, pp. 365–394] as a guiding reference. Accordingly, we proceed as follows:

Remark 2.2. *Defining*

$$\widetilde{Z}(x_1,...,x_N) := \widetilde{Z}(x) := \prod_{i=1}^{N} W(x_i) = \frac{1}{2^N}\prod_{i=1}^{N}\left(G_{q,\lambda} + G_{\frac{1}{q},\lambda}\right)(x_i),$$

$x = (x_1,...,x_N) \in \mathbb{R}^N$, $\lambda, q > 0$, $N \in \mathbb{N}$, *we get:*

(i)
$$\widetilde{Z}(x) > 0, \ \forall\ x \in \mathbb{R}^N,$$

(ii)
$$\sum_{k=-\infty}^{\infty} \widetilde{Z}(x-k) := \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} \widetilde{Z}(x_1-k_1,...,x_N-k_N) = 1,$$

$k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall\ x \in \mathbb{R}^N$,

(iii)
$$\sum_{k=-\infty}^{\infty} \widetilde{Z}(nx-k) = 1, \tag{2.3}$$

$\forall\ x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

(iv)
$$\int_{\mathbb{R}^N} \widetilde{Z}(x)\,dx = 1,$$

that is $\widetilde{Z}$ *is a multivariate density function.*

We denote $\|x\|_\infty := \max\{|x_1|,...,|x_N|\}$, $x \in \mathbb{R}^N$, $\infty := (\infty,...,\infty)$, $-\infty := (-\infty,...,-\infty)$, $\lceil na\rceil := (\lceil na_1\rceil,...,\lceil na_N\rceil)$, $\lfloor nb\rfloor := (\lfloor nb_1\rfloor,...,\lfloor nb_N\rfloor)$, $a := (a_1,...,a_N)$, $b := (b_1,...,b_N)$.

We immediately observe that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \widetilde{Z}(nx-k) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N} W(nx_i-k_i)\right) = \prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} W(nx_i-k_i)\right).$$

(v) *Then we have*

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} \widetilde{Z}(nx-k) < \widetilde{\gamma} A^{-\lambda\left(n^{1-\beta}-2\right)}, \quad \textit{where } 0<\beta<1,$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N}[a_i,b_i]$.

(vi) *We get the next inequality:*

$$0 < \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \widetilde{Z}(nx-k)} < (K(q))^N, \tag{2.4}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n \in \mathbb{N}$.

It is also easy to see that

(vii)

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} \widetilde{Z}(nx-k) < \widetilde{\gamma} A^{-\lambda\left(n^{1-\beta}-2\right)}, \tag{2.5}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widetilde{Z}(nx-k) \neq 1,$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

Here $\left(X, \|\cdot\|_{\gamma}\right)$ *is a Banach space.*

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$, $x = (x_1, ..., x_N) \in \prod_{i=1}^{N} [a_i, b_i]$, $n \in \mathbb{N} : \lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the following multivariate linear normalized symmetrized neural network operator, let $x := (x_1, ..., x_N) \in \prod_{i=1}^{N} [a_i, b_i]$*:*

$$\begin{aligned} \Psi_n^s(f, x_1, ..., x_N) &:= \Psi_n^s(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \widetilde{Z}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widetilde{Z}(nx-k)} \\ &= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \left(G_{q,\lambda}(nx_i - k_i) + G_{\frac{1}{q},\lambda}(nx_i - k_i)\right)\right)} \\ &\times \left(\prod_{i=1}^{N} \left(G_{q,\lambda}(nx_i - k_i) + G_{\frac{1}{q},\lambda}(nx_i - k_i)\right)\right). \end{aligned}$$

For large enough $n \in \mathbb{N}$ *we always obtain* $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. *Also* $a_i \leq \frac{k_i}{n} \leq b_i$, *iff* $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $g \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ *we define the companion operator*

$$\widetilde{\Psi}_n^s(g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) \widetilde{Z}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widetilde{Z}(nx-k)}. \tag{2.6}$$

Clearly $\widetilde{\Psi}_n^s$ *is a positive linear operator. We have that*

$$\widetilde{\Psi}_n^s(1, x) = 1, \ \ \forall\, x \in \prod_{i=1}^{N} [a_i, b_i].$$

Notice that $\Psi_n^s(f) \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$ *and* $\widetilde{\Psi}_n^s(g) \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

We also get

$$\left\| \Psi_n^s(f, x) \right\|_{\gamma} \leq \widetilde{\Psi}_n^s\left(\|f\|_{\gamma}, x\right), \tag{2.7}$$

$\forall\, x \in \prod_{i=1}^{N} [a_i, b_i]$, *and*

$$\Psi_n^s(cg,x) = c\widetilde{\Psi}_n^s(g,x), \quad \forall\ x \in \prod_{i=1}^N [a_i,b_i], \tag{2.8}$$

and

$$\Psi_n^s(c) = c, \quad \forall\ c \in X. \tag{2.9}$$

Set also

$$^*\Psi_n^s(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \widetilde{Z}(nx-k). \tag{2.10}$$

Consequently we derive (*by* (2.4))

$$|\Psi_n^s(f,x) - f(x)| \le (K(q))^N \left|^*\Psi_n^s(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widetilde{Z}(nx-k)\right|, \tag{2.11}$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i,b_i]\right).$

We will estimate the right hand side of (2.11)*, when* $X = (\mathbb{C}, |\cdot|)$*, the complex numbers.*

For the last and others we need

Definition 2.1. (see [5, p. 274]) *Let* M *be a convex and compact subset of* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$*,* $p \in [1,\infty]$*, and* $\left(X, \|\cdot\|_\gamma\right)$ *be a Banach space. Let* $f \in C(M,X)$*. We define the first modulus of continuity of* f *as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \le \delta}} \|f(x) - f(y)\|_\gamma, \quad 0 < \delta \le diam(M). \tag{2.12}$$

If $\delta > diam(M)$*, then*

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{2.13}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 2.1. (see [5, p. 274]) *We have* $\omega_1(f,\delta) \to 0$ *as* $\delta \downarrow 0$*, iff* $f \in C(M,X)$*, where* M *is a convex compact subset of* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$*,* $p \in [1,\infty]$.

3 Abstract Korovkin Inequalities

All here in this section come from [4] and [5], Chapter 10, pp. 261-281.

We give

Remark 3.1. *Let* $\left(\mathbb{R}^k, \|\cdot\|_p\right)$*,* $k \in \mathbb{N}$*; where* $\|\cdot\|_p$ *is the* L_p*-norm,* $1 \le p \le \infty$*.* $\mathbb{R}^k$ *is a Banach space, and* $\left(\mathbb{R}^k\right)^j$ *denotes the* j*-fold product space* $\mathbb{R}^k \times ... \times \mathbb{R}^k$ *endowed with the max-norm* $\|x\|_{\left(\mathbb{R}^k\right)^j} := \max_{1 \le \lambda \le j} \|x_\lambda\|_p$*, where* $x := (x_1, ..., x_j) \in \left(\mathbb{R}^k\right)^j$.

Let $\left(X, \|\cdot\|_{\gamma}\right)$ *be a general Banach space. Then the space* $L_j := L_j\left(\left(\mathbb{R}^k\right)^j ; X\right)$ *of all* j*-multilinear continuous maps* $g : \left(\mathbb{R}^k\right)^j \to X$, $j = 1, ..., n$, *is a Banach space with norm*

$$\|g\| := \|g\|_{L_j} := \sup_{\|x\|_{\left(\mathbb{R}^k\right)^j}=1} \|g(x)\|_{\beta} = \sup \frac{\|g(x)\|_{\beta}}{\|x_1\|_p \dots \|x_j\|_p}.$$

Let M *be a non-empty convex and compact subset of* $\mathbb{R}^k$ *and* $x_0 \in M$ *is fixed.*

Let O *be an open subset of* $\mathbb{R}^k : M \subset O$. *Let* $f : O \to X$ *be a continuous function whose Fréchet derivatives* (see [12]) $f^{(j)} : O \to L_j = L_j\left(\left(\mathbb{R}^k\right)^j ; X\right)$ *exist and are continuous for* $1 \le j \le n$, $n \in \mathbb{N}$. *Call* $(x - x_0)^j := (x - x_0, ..., x - x_0) \in \left(\mathbb{R}^k\right)^j$, $x \in M$.

Here we deal with $f|_M$.

Then by Taylor's formula (see [8] and [12, p. 124]), *we get*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_n(x, x_0), \quad \text{all } x \in M,$$

where the remainder is the Riemann integral

$$R_n(x, x_0) := \int_0^1 \frac{(1-u)^{n-1}}{(n-1)!} \left(f^{(n)}(x_0 + u(x - x_0)) - f^{(n)}(x_0)\right)(x - x_0)^n du,$$

here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

Considering

$$w := \omega_1\left(f^{(n)}, h\right) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \le h}} \left\|f^{(n)}(x) - f^{(n)}(y)\right\|,$$

we obtain

$$\begin{aligned}
& \left\|\left(f^{(n)}(x_0 + u(x - x_0)) - f^{(n)}(x_0)\right)(x - x_0)^n\right\|_{\gamma} \\
\le\ & \left\|f^{(n)}(x_0 + u(x - x_0)) - f^{(n)}(x_0)\right\| \cdot \|x - x_0\|_p^n \\
\le\ & w \cdot \|x - x_0\|_p^n \cdot \left\lceil \frac{u\|x - x_0\|_p}{h} \right\rceil,
\end{aligned}$$

by Lemma 7.1.1 of [2, p. 208].

Therefore for all $x \in M$ (see [2, pp. 121-122]):

$$\|R_n(x, x_0)\|_{\beta} \le w \|x - x_0\|_p^n \int_0^1 \left\lceil \frac{u\|x - x_0\|_p}{h} \right\rceil \frac{(1-u)^{n-1}}{(n-1)!} du$$

$$= w\Phi_n\left(\|x - x_0\|_p\right)$$

by a change of variable, where

$$\Phi_n(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t| - s)^{n-1}}{(n-1)!} ds = \frac{1}{n!}\left(\sum_{j=0}^{\infty} (|t| - jh)_+^n\right), \quad \forall\, t \in \mathbb{R},$$

is a (polynomial) spline function, see [2, pp. 210-211]. *Also from there we get*

$$\Phi_n(t) \leq \left(\frac{|t|^{n+1}}{(n+1)!h} + \frac{|t|^n}{2n!} + \frac{h|t|^{n-1}}{8(n-1)!} \right), \quad \forall\, t \in \mathbb{R},$$

with equality true only at $t = 0$. *Therefore it holds*

$$\|R_n(x, x_0)\|_\gamma \leq w \left(\frac{\|x - x_0\|_p^{n+1}}{(n+1)!h} + \frac{\|x - x_0\|_p^n}{2n!} + \frac{h\|x - x_0\|_p^{n-1}}{8(n-1)!} \right), \quad \forall\, x \in M.$$

We have found that

$$\left\| f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} \right\|_\gamma$$
$$\leq \omega_1\left(f^{(n)}, h\right) \left(\frac{\|x - x_0\|_p^{n+1}}{(n+1)!h} + \frac{\|x - x_0\|_p^n}{2n!} + \frac{h\|x - x_0\|_p^{n-1}}{8(n-1)!} \right) < \infty, \tag{3.1}$$

$\forall\, x, x_0 \in M$. *Here* $0 < \omega_1\left(f^{(n)}, h\right) < \infty$, *by* M *being compact and* $f^{(n)}$ *being continuous on* M.

One can rewrite (3.1) *as*

$$\left\| f(\cdot) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \leq \omega_1\left(f^{(n)}, h\right) \left(\frac{\|\cdot - x_0\|_p^{n+1}}{(n+1)!h} + \frac{\|\cdot - x_0\|_p^n}{2n!} + \frac{h\|\cdot - x_0\|_p^{n-1}}{8(n-1)!} \right),$$

$(\forall x_0 \in M)$, *a pointwise functional inequality on* M. *Here* $(\cdot - x_0)^j$ *maps* M *into* $\left(\mathbb{R}^k\right)^j$ *and it is continuous, also* $f^{(j)}(x_0)$ *maps* $\left(\mathbb{R}^k\right)^j$ *into* X *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot - x_0)^j$ *is continuous from* M *into* X.

Clearly $f(\cdot) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)$, *hence* $\left\| f(\cdot) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \in C(M)$.

We mention the following fundamental approximation result.

Theorem 3.1. *Let* M *be a nonempty compact convex subset of the open subset* O *of* $\left(\mathbb{R}^k, \|\cdot\|_p\right)$, $1 \leq p \leq \infty$, *and let* $\left(X, \|\cdot\|_\gamma\right)$ *be a general Banach space. For any* $N \in \mathbb{N}$ *let the linear operators* $L_N : C(M, X) \to C(M, X)$ *and the positive linear operators* $\widetilde{L}_N : C(M) \to C(M)$, *such that*

$$\|(L_N(f))(x_0)\|_\gamma \leq \left(\widetilde{L}_N\left(\|f\|_\gamma\right)\right)(x_0),$$

$\forall\, N \in \mathbb{N}$, $\forall\, f \in C(M, X)$, $\forall\, x_0 \in M$. *Furthermore assume that*

$$L_N(cF) = c\widetilde{L}_N(F), \quad \forall\, F \in C(M), \forall\, c \in X.$$

Let $n \in \mathbb{N}$, *here we deal with* $f \in C^n(O, X)$, *the space of* n*-times continuously Fréchet differentiable functions from* O *into* X.

Here we study approximation to $f|_M$.

Let $x, x_0 \in M$, $r > 0$, *and*

$$\omega_1\left(f^{(n)}, h\right) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \leq h}} \left\| f^{(n)}(x) - f^{(n)}(y) \right\|.$$

Then

(1)

$$\begin{aligned}
&\left\| \left(L_N(f)\right)(x_0) - \sum_{j=0}^{n} \frac{1}{j!} \left(\widetilde{L}_N \left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0) \right\|_{\gamma} \\
\leq\ & \frac{1}{rn!}\omega_1\left(f^{(n)}, r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\left(\frac{n}{n+1}\right)} \\
\times\ & \left[\frac{1}{(n+1)} + \frac{r}{2}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}} + \frac{nr^2}{8}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}\right],
\end{aligned}$$

(2) *additionally if* $f^{(j)}(x_0) = 0$, $j = 0, 1, ..., n$, *we have*

$$\begin{aligned}
&\left\|\left(L_N(f)\right)(x_0)\right\|_{\gamma} \\
\leq\ & \frac{1}{rn!}\omega_1\left(f^{(n)}, r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\left(\frac{n}{n+1}\right)} \\
\times\ & \left[\frac{1}{(n+1)} + \frac{r}{2}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}} + \frac{nr^2}{8}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}\right],
\end{aligned}$$

(3)

$$\begin{aligned}
&\left\|\left(L_N(f)\right)(x_0) - f(x_0)\right\|_{\gamma} \\
\leq\ & \|f(x_0)\| \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \sum_{j=1}^{n} \frac{1}{j!} \left\|\left(\widetilde{L}_N\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_{\gamma} \\
+\ & \frac{1}{rn!}\omega_1\left(f^{(n)}, r\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)\left(\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right)^{\left(\frac{n}{n+1}\right)} \\
\times\ & \left[\frac{1}{(n+1)} + \frac{r}{2}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{n+1}} + \frac{nr^2}{8}\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{2}{n+1}}\right], \qquad (3.2)
\end{aligned}$$

(4)

$$\begin{aligned}
\left\|\left\|L_N(f) - f\right\|_{\gamma}\right\|_{\infty,M} \leq\ & \left\|\|f\|_{\gamma}\right\|_{\infty,M} \left\|\left(\widetilde{L}_N(1)\right) - 1\right\|_{\infty,M} \\
+\ & \sum_{j=1}^{n} \frac{1}{j!} \left\|\left\|\left(\widetilde{L}_N\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_{\gamma}\right\|_{\infty, x_0 \in M} \\
+\ & \frac{1}{rn!}\omega_1\left(f^{(n)}, r\left\|\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in M}^{\frac{1}{n+1}}\right) \\
\times\ & \left\|\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{n+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in M}^{\left(\frac{n}{n+1}\right)} \\
\times\ & \left[\frac{1}{(n+1)} + \frac{r}{2}\left\|\widetilde{L}_N(1)\right\|_{\infty,M}^{\frac{1}{n+1}} + \frac{nr^2}{8}\left\|\widetilde{L}_N(1)\right\|_{\infty,M}^{\frac{2}{n+1}}\right]. \qquad (3.3)
\end{aligned}$$

We need

Lemma 3.1. *The function* $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^m\right)\right)(x_0)$ *is continuous at* $x_0 \in M$, $m \in \mathbb{N}$.

Remark 3.2. *We also have*

$$\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^k\right)\right)(x_0)\right\|_{\infty,x_0\in M}\leq\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right\|_{\infty,x_0\in M}^{\left(\frac{k}{n+1}\right)}\left\|\widetilde{L}_N(1)\right\|_{\infty,M}^{\left(\frac{n+1-k}{n+1}\right)},\tag{3.4}$$

for all $k=1,...,n$.

Conclusion 3.1. *Let* $N\to\infty$ *and* $\widetilde{L}_N(1)\xrightarrow{u}1$*, uniformly, and* $\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right\|_{\infty,x_0\in M}\to 0$*. Then by* (3.3) *we get, as* $N\to\infty$*, that* $L_n(f)\xrightarrow{u}f$*, uniformly in* $\|\cdot\|_\gamma$ *over* M.

The last statement is also supported by

$$\left\|\left(L_N\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\beta\leq\left\|f^{(j)}\right\|\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^j\right)\right)(x_0)\right\|_{\infty,x_0\in M}$$

and (3.4)*. Here notice that* $\left\|\widetilde{L}_N(1)\right\|_{\infty,M}$ *turns out to be bounded, and* $\omega_1\left(f^{(n)},\cdot\right)$ *is also bounded.*

Conclusion 3.2. *By* (3.2) *we get the following: as* $N\to\infty$*, assume that* $\left(\widetilde{L}_N(1)\right)(x_0)\to 1$*, and* $\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{n+1}\right)\right)(x_0)\right)\to 0$*. Then* $(L_N(f))(x_0)\to f(x_0)$*, as* $N\to\infty$*, pointwise. Notice that here* $\left(\widetilde{L}_N(1)\right)(x_0)$ *is bounded with respect to* N*, as well as* $\omega_1\left(f^{(n)},\cdot\right)$ *is bounded in general.*

Corollary 3.1. (*to Theorem* 3.1*, case* $n=1$*, see* (3.2)) *Additionally assume that* $\left(\widetilde{L}_N(1)\right)(x_0)=1$*,* $\forall\ x_0\in M$*. Then*

(1)

$$\begin{aligned}\|(L_N(f))(x_0)-f(x_0)\|_\gamma&\leq\left\|\left(\widetilde{L}_N\left(f^{(1)}(x_0)(\cdot-x_0)\right)\right)(x_0)\right\|_\gamma\\&+\frac{1}{2r}\omega_1\left(f^{(1)},r\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right)\\&\times\left(\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\left[1+r+\frac{r^2}{4}\right]\end{aligned}$$

(2)

$$\begin{aligned}\|\|(L_N(f))-f\|\|_{\infty,M}&\leq\left\|\left\|\left(\widetilde{L}_N\left(f^{(1)}(x_0)(\cdot-x_0)\right)\right)(x_0)\right\|_\gamma\right\|_{\infty,x_0\in M}\\&+\frac{1}{2r}\omega_1\left(f^{(1)},r\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right\|_{\infty,x_0\in M}^{\frac{1}{2}}\right)\\&\times\left\|\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right\|_{\infty,x_0\in M}^{\frac{1}{2}}\left[1+r+\frac{r^2}{4}\right].\end{aligned}$$

We need

Lemma 3.2. (*see* [2, p. 208]) *Let* M *be a convex compact subset of* $\left(\mathbb{R}^k,\|\cdot\|_p\right)$*,* $p\in[1,\infty]$*, and* $\left(X,\|\cdot\|_\gamma\right)$ *be a general Banach space. Let* $f\in B(M,X)$ *(bounded functions). Then*

$$\|f(x)-f(x_0)\|_\gamma\leq\omega_1(f,h)\left\lceil\frac{\|x-x_0\|_p}{h}\right\rceil\leq\omega_1(f,h)\left(1+\frac{\|x-x_0\|_p}{h}\right),\quad\forall\ x\in M,\tag{3.5}$$

where $h>0$.

We present the basic pointwise convergence result

Theorem 3.2. *Let $L_N : C(M,X) \to C(M,X)$ be linear operators, $\forall\ N \in \mathbb{N}$; where M is a convex compact subset of $\left(\mathbb{R}^k, \|\cdot\|_p\right)$, $p \in [1,\infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ is a general Banach space. Let $\widetilde{L}_N : C(M) \to C(M)$ be positive linear operators, such that*

$$\|(L_N(f))(x_0)\|_\gamma \le \left(\widetilde{L}_N\left(\|f\|_\gamma\right)\right)(x_0),$$

$\forall\ N \in \mathbb{N}$, $\forall\ f \in C(M,X)$, $\forall\ x_0 \in M$. Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\ g \in C(M), \forall\ c \in X.$$

Then

$$\begin{aligned} \|(L_N(f))(x_0) - f(x_0)\|_\gamma &\le \|f(x_0)\|_\gamma \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| \\ &+ \left[\left(\widetilde{L}_N(1)\right)(x_0) + 1\right]\omega_1\left(f, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right), \end{aligned} \tag{3.6}$$

$\forall f \in C(M,X)$. By (78), as $\left(\widetilde{L}_N(1)\right)(x_0) \to 1$ and $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) \to 0$, then $(L_N(f))(x_0) \overset{\|\cdot\|_\gamma}{\to} f(x_0)$, as $N \to \infty$, $\forall\ f \in C(M,X)$. Here notice that $\left(\widetilde{L}_N(1)\right)(x_0)$ is bounded.

Corollary 3.2. *All as in Theorem 3.2. Additionally assume that $\left(\widetilde{L}_N(1)\right)(x_0) = 1$, $\forall\ x_0 \in M$. Then*

$$\|(L_N(f))(x_0) - f(x_0)\|_\gamma \le 2\omega_1\left(f, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right), \quad \forall\ f \in C(M,X). \tag{3.7}$$

Next we obtain the following uniform convergence result.

Theorem 3.3. *All as in Theorem 3.2. Then*

$$\begin{aligned} \left\|\|L_N(f) - f\|_\gamma\right\|_{\infty,M} &\le \left\|\|f\|_\gamma\right\|_{\infty,M} \left\|\widetilde{L}_N(1) - 1\right\|_{\infty,M} \\ &+ \left\|\widetilde{L}_N(1) + 1\right\|_{\infty,M} \\ &\times\ \omega_1\left(f, \left\|\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right\|_{\infty,x_0\in M}\right), \end{aligned} \tag{3.8}$$

$\forall f \in C(M,X)$.

Conclusion 3.3. *By Lemma 3.1 $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)$ is continuous at $x_0 \in M$. Thus $\left\|\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right\|_{\infty,x_0\in M} < \infty$. As $N \to \infty$, we assume that $\widetilde{L}_N(1) \overset{u}{\to} 1$, uniformly, and $\left\|\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right\|_{\infty,x_0\in M} \to 0$, then $L_n(f) \overset{u}{\to} f$, uniformly on M, $\forall\ f \in C(M,X)$. Here $\left\|\widetilde{L}_N(1)\right\|$ is bounded.*

Under convexity we have the following sharp general pointwise convergence result. Here M^0 denotes the interior of M.

Theorem 3.4. *All here as in Theorem 3.2. Additionally, assume that* $x_0 \in M^0$, *and the closed ball in* $\mathbb{R}^k : B\left(x_0, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right) \subset M$, *where* $\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0) \geq 0$, *and* $\|f(t) - f(x_0)\|_\gamma$ *is convex in* $t \in M$. *Then*

$$\|(L_N(f))(x_0) - f(x_0)\|_\gamma \leq \|f(x_0)\|_\gamma \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \omega_1\left(f, \left(\widetilde{L}_N\left(\|\cdot - x_0\|_p\right)\right)(x_0)\right), \quad (3.9)$$

$\forall\ N \in \mathbb{N}$.

Theorem 3.5. *All as in Theorem 3.4. Inequality (3.9) is sharp, in fact it is attained by* $f(t) = \vec{i}\,\|t - x_0\|_p$, $\vec{i}$ *is a unit vector of* $\left(X, \|\cdot\|_\gamma\right)$, $t \in M$.

4 Main Results

In this section we apply the results of Section 3 to the linear neural network operators Ψ_n^s and $\widetilde{\Psi}_n^s$, $n \in \mathbb{N}$, of Section 2. In fact $\widetilde{\Psi}_n^s$ are positive linear neural network operators, with the property $\widetilde{\Psi}_n^s(1) = 1$. Also (2.7)–(2.11) are valid.

From now on $M = \prod_{i=1}^{N} [a_i, b_i] \subset \left(\mathbb{R}^N, \|\cdot\|_p\right)$, $N \in \mathbb{N}$, where $1 \leq p \leq \infty$.

We present the following general approximation result.

Theorem 4.1. *Let* $\left(X, \|\cdot\|_\gamma\right)$ *be a general Banach space,* $x_0 \in \prod_{i=1}^{N} [a_i, b_i]$, $1 \leq p \leq \infty$. *Let* $m \in \mathbb{N}$, $f \in C^m\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$, *the space of m-times continuously Fréchet differentiable functions from* $\prod_{i=1}^{N} [a_i, b_i]$ *into* X. *Let* $x, x_0 \in \prod_{i=1}^{N} [a_i, b_i]$, $r > 0$, *and*

$$\omega_1\left(f^{(m)}, h\right) := \sup_{\substack{x,y \in \prod_{i=1}^{N} [a_i,b_i]: \\ \|x-y\|_p \leq h}} \left\|f^{(m)}(x) - f^{(m)}(y)\right\|, \quad h > 0.$$

Then

(1)

$$\left\|(\Psi_n^s(f))(x_0) - \sum_{j=0}^{n} \frac{1}{j!}\left(\widetilde{\Psi}_n^s\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma$$
$$\leq \frac{1}{rm!}\omega_1\left(f^{(m)}, r\left(\left(\widetilde{\Psi}_n^s\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)$$
$$\times \left(\left(\widetilde{\Psi}_n^s\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

(2) *additionally if* $f^{(j)}(x_0) = 0$, $j = 0, 1, ..., m$, *we have*

$$\|(\Psi_n^s(f))(x_0)\|_\gamma$$
$$\leq \frac{1}{rm!}\omega_1\left(f^{(m)}, r\left(\left(\widetilde{\Psi}_n^s\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)$$
$$\times \left(\left(\widetilde{\Psi}_n^s\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

(3)

$$\begin{aligned}
&\left\| \left(\Psi_n^s (f) \right) (x_0) - f(x_0) \right\|_\gamma \leq \sum_{j=1}^{n} \frac{1}{j!} \left\| \left(\widetilde{\Psi}_n^s \left(f^{(j)} (x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \\
+ \;& \frac{1}{rm!} \omega_1 \left(f^{(m)}, r \left(\left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^{m+1} \right) \right) (x_0) \right)^{\frac{1}{m+1}} \right) \\
\times \;& \left(\left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^{m+1} \right) \right) (x_0) \right)^{\left(\frac{m}{m+1} \right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \qquad (4.1)
\end{aligned}$$

(4)

$$\begin{aligned}
& \left\| \left\| \Psi_n^s (f) (x_0) - f(x_0) \right\|_\gamma \right\|_{\infty, \prod_{i=1}^{N} [a_i, b_i]} \\
\leq \;& \sum_{j=1}^{n} \frac{1}{j!} \left\| \left\| \left(\widetilde{\Psi}_n^s \left(f^{(j)} (x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} \\
+ \;& \frac{1}{rm!} \omega_1 \left(f^{(m)}, r \left\| \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^{m+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{m+1}} \right) \\
\times \;& \left\| \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^{m+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\left(\frac{m}{m+1} \right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right]. \qquad (4.2)
\end{aligned}$$

Proof. A direct application of Theorem 3.1. □

We state

Remark 4.1. *Clearly here $\omega_1 \left(f^{(m)}, \cdot \right)$ is bounded. By Lemma 3.1 we get that $\left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^{\lambda} \right) \right) (x_0)$ is continuous at $x_0 \in \prod_{i=1}^{N} [a_i, b_i]$, $\lambda \in \mathbb{N}$. And, by (3.4) we obtain*

$$\left\| \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^{k} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} \leq \left\| \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^{m+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\left(\frac{k}{m+1} \right)}, \qquad (4.3)$$

for $k = 1, \ldots, m$.

We give

Remark 4.2. *Here we take $p=\infty$. We observe that*

$$
\begin{aligned}
& \left(\widetilde{\Psi}_n^s\left(\|\cdot-x_0\|_\infty^{m+1}\right)\right)(x_0) \\
\overset{(2.6)}{=} & \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^{m+1}\widetilde{Z}(nx_0-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\widetilde{Z}(nx_0-k)} \\
\overset{(2.4)}{\leq} & (K(q))^N\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^{m+1}\widetilde{Z}(nx_0-k)\right) \\
= & (K(q))^N\left(\sum_{\begin{cases} k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x_0\right\|_\infty\leq\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^{m+1}\widetilde{Z}(nx_0-k)\right. \\
+ & \left.\sum_{\begin{cases} k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x_0\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^{m+1}\widetilde{Z}(nx_0-k)\right) \\
\overset{((2.3),(2.5))}{\leq} & (K(q))^N\left[\frac{1}{n^{\beta(m+1)}}+\|b-a\|_\infty^{m+1}\widetilde{\gamma}A^{-\lambda\left(n^{1-\beta}-2\right)}\right].
\end{aligned}
$$

We have proved that

$$
\begin{aligned}
& \left\|\left(\widetilde{\Psi}_n^s\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^N[a_i,b_i]} \\
\leq & (K(q))^N\left[\frac{1}{n^{\beta(m+1)}}+\|b-a\|_\infty^{m+1}\widetilde{\gamma}A^{-\lambda\left(n^{1-\beta}-2\right)}\right]\to 0,
\end{aligned}
\tag{4.4}
$$

as $n\to\infty$. Here it is $0<\beta<1$, $\widetilde{\gamma}:=2\max\left\{q,\frac{1}{q}\right\}$, $A>1$; $n\in\mathbb{N}:n^{1-\beta}>2$, $x_0\in\prod_{i=1}^N[a_i,b_i]$.

So, by Conclusion 3.1, see also (4.2) and (4.3), we derive that $\Psi_n^s(f)\xrightarrow{u}f$, uniformly in $\|\cdot\|_\gamma$ over $\prod_{i=1}^N[a_i,b_i]$, as $n\to\infty$; $p=\infty$ case. Pointwise convergence is obviously valid too.

We continue with

Corollary 4.1. (*to Theorem 4.1, case of $m=1$, see (4.1)*) *Let $x_0\in\prod_{i=1}^N[a_i,b_i]$, then*

(1)

$$
\begin{aligned}
\left\|\left(\Psi_n^s(f)\right)(x_0)-f(x_0)\right\|_\gamma \leq & \left\|\left(\widetilde{\Psi}_n^s\left(f^{(1)}(x_0)(\cdot-x_0)\right)\right)(x_0)\right\|_\gamma \\
+ & \frac{1}{2r}\omega_1\left(f^{(1)},r\left(\left(\widetilde{\Psi}_n^s\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right) \\
\times & \left(\left(\widetilde{\Psi}_n^s\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\left[1+r+\frac{r^2}{4}\right],
\end{aligned}
$$

and

(2)

$$\begin{aligned} & \left\| \left\| \left(\Psi_n^s(f) \right)(x_0) - f(x_0) \right\|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i,b_i]} \\ \leq & \left\| \left\| \left(\widetilde{\Psi}_n^s \left(f^{(1)}(x_0)(\cdot - x_0) \right) \right)(x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i,b_i]} \\ + & \frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i,b_i]}^{\frac{1}{2}} \right) \\ \times & \left\| \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i,b_i]}^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right]. \end{aligned} \tag{4.5}$$

Proof. Direct application to Corollary 3.1. □

We give

Remark 4.3. *By (4.4) we get that*

$$\left\| \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i,b_i]} \leq (K(q))^N \left[\frac{1}{n^{2\beta}} + \|b-a\|_\infty^2 \widetilde{\gamma} A^{-\lambda\left(n^{1-\beta}-2\right)} \right] \to 0,$$

as $n \to \infty$. Again by (4.5) we obtain $\Psi_n^s(f) \to f$, as $n \to \infty$, pointwise and uniformly, for $p = \infty$ case.

Note 4.1. *For $f \in C\left(\prod_{i=1}^N [a_i,b_i], X\right)$ the modulus of continuity $\omega_1(f,\delta)$ is defined by (2.12), (2.13). By Lemma 2.1, we have that $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$ iff $f \in C\left(\prod_{i=1}^N [a_i,b_i], X\right)$. Also Lemma 3.2 now applies for an $f \in B\left(\prod_{i=1}^N [a_i,b_i], X\right)$ and (3.5) is valid.*

We give

Theorem 4.2. *Let $f \in C\left(\prod_{i=1}^N [a_i,b_i], X\right)$, where $1 \leq p \leq \infty$.*

Then

(1)

$$\left\| \left(\Psi_n^s(f) \right)(x_0) - f(x_0) \right\|_\gamma \leq 2\omega_1 \left(f, \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p \right) \right)(x_0) \right),$$

and

(2)

$$\left\| \left\| \left(\Psi_n^s(f) \right)(x_0) - f(x_0) \right\|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i,b_i]} \leq 2\omega_1 \left(f, \left\| \left(\widetilde{\Psi}_n^s \left(\| \cdot - x_0 \|_p \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i,b_i]} \right), \tag{4.6}$$

$\forall\ f \in C\left(\prod_{i=1}^N [a_i,b_i], X\right)$.

Proof. Direct application of (3.6), (3.7), (3.8). □

We state

Remark 4.4. *Inequality (4.4) is clearly valid for $m=0$. So we have*

$$\left\| \left\| \left(\Psi_n^s (f) \right) (x_0) - f(x_0) \right\|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i,b_i]} \leq (K(q))^N \left[\frac{1}{n^\beta} + \|b-a\|_\infty \widetilde{\gamma} A^{-\lambda\left(n^{1-\beta}-2\right)} \right] \to 0, \tag{4.7}$$

as $n \to \infty$. By (4.6) now we obtain $\Psi_n^s(f) \to f$, as $n \to \infty$, pointwise and uniformly, for $p=\infty$ case.

Corollary 4.2. *It holds $(0<\beta<1,\ p=\infty)$*

$$\left\| \left\| \left(\Psi_n^s (f) \right) (x_0) - f(x_0) \right\|_\gamma \right\|_{\infty, \prod\limits_{i=1}^N [a_i,b_i]} \leq 2\omega_1 \left(f, (K(q))^N \left[\frac{1}{n^\beta} + \|b-a\|_\infty \widetilde{\gamma} A^{-\lambda\left(n^{1-\beta}-2\right)} \right] \right),$$

$\forall\ f \in C\left(\prod_{i=1}^N [a_i,b_i], X \right)$.

Proof. By (4.6), (4.7), $p=\infty$. ☐

Under convexity we derive the following result.

Theorem 4.3. *Let $f \in C\left(\prod_{i=1}^N [a_i,b_i], X \right)$, $1 \leq p \leq \infty$. Here $x_0 \in \left(\prod_{i=1}^N [a_i,b_i] \right)^0$ and $\|f(t) - f(x_0)\|_\gamma$ is convex in $t \in \prod_{i=1}^N [a_i,b_i]$. We assume that the closed ball in $\mathbb{R}^N$: $B\left(x_0, \left(\widetilde{\Psi}_n^s \left(\|\cdot - x_0\|_p \right) \right)(x_0) \right) \subset \prod_{i=1}^N [a_i,b_i]$. Then*

$$\left\| \left(\Psi_n^s (f) \right) (x_0) - f(x_0) \right\|_\gamma \leq \omega_1 \left(f, \left(\widetilde{\Psi}_n^s \left(\|\cdot - x_0\|_p \right) \right)(x_0) \right), \tag{4.8}$$

$\forall\ N \in \mathbb{N}, n^{1-\beta} > 2,\ 0<\beta<1$.

Proof. By Theorem 3.4. Notice here that $\left(\widetilde{\Psi}_n^s \left(\|\cdot - x_0\|_p \right) \right)(x_0) > 0$. ☐

Theorem 4.4. *All as in Theorem 4.3. Inequality (4.8) is sharp, in fact it is attained by $f(t) = \vec{i}\ \|t-x_0\|_p$, $\vec{i}$ is a unit vector of $\left(X, \|\cdot\|_\gamma \right)$, $t \in \prod_{i=1}^N [a_i,b_i]$.*

Proof. By Theorem 3.5. ☐

We finish with

Corollary 4.3. *All as in Theorem 4.3, $p=\infty$ case. Then*

$$\left\| \left(\Psi_n^s (f) \right) (x_0) - f(x_0) \right\|_\gamma \leq \omega_1 \left(f, (K(q))^N \left[\frac{1}{n^\beta} + \|b-a\|_\infty \widetilde{\gamma} A^{-\lambda\left(n^{1-\beta}-2\right)} \right] \right),$$

$\forall\ N \in \mathbb{N} : n^{1-\beta} > 2,\ 0<\beta<1$.

Clearly, $\lim\limits_{n\to\infty} \Psi_n^s (f)(x_0) = f(x_0)$, at $x_0 \in \left(\prod_{i=1}^N [a_i,b_i] \right)^0$ as in the assumptions.

5 Conclusion and Future Directions

In this paper, a new theoretical framework has been proposed by examining multivariate abstract neural network operators through the lens of positive linear operators. This perspective makes it possible to apply well-established techniques from approximation theory to neural network structures defined via finite summations. As a result, both pointwise and uniform approximation properties have been derived under this new setting.

A notable aspect of the study is the connection established between neural networks and measure theory through the use of the Riesz representation theorem. This adds depth to the analytical foundation of neural networks and introduces a measure-theoretic view to their approximation capabilities. Additionally, the consideration of convexity in the analysis allowed us to derive more refined and, in some cases, optimal results.

Looking ahead, there are several promising directions for further research. One natural extension would be to investigate broader classes of activation functions and study how the structure of the network influences convergence behavior. Exploring these operators in different normed spaces or under alternative topological assumptions could also provide valuable insights. Moreover, introducing stochastic elements or probabilistic frameworks may help bridge the gap between this abstract approach and more practical learning scenarios.

Overall, this work initiates a connection between two seemingly distant areas, namely neural network approximation and classical positive operator theory, and offers a foundation for deeper mathematical exploration in future studies.

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